Lecture 3: Introduction to Discrete Choice

- Choice Theory
- Utility Theory

Choice Theory

- In every choice process
  - The decision-maker
  - The alternatives
  - Attributes of alternatives
  - Decision rule

The Alternatives (the choice set)

- Must be mutually exclusive
  - The decision maker choose only one alternative
- Must be exhaustive
  - All possible alternative are included
- The number of alternatives must be finite

Utility Theory

- Preferences
- Utility maximization
- Utility function
  - Has a numerical value which depends on attributes of the available options and the individual
  - Its value for one option exceeds its value for another if and only if the individual prefers the first option to the second
  - The individual chooses the most preferred option, which is the one with the highest utility-function value

Mathematical Presentation

c = The choice set: the set of options available to an individual

i = index for options •

x_i = attributes of option *i* –

s = attributes of the individual •

For any two options *i* and *j* in c, –

U(x1,s)>U(x,,s)

implies that the individual prefers alternative *i* to alternative *j*

A Utility Model of Mode Choice

Example

U = -T-5C/Y

Mode	Time (hours)	Cost ($)	Income y=40	Income y=10
Drive Alone	0.50	2.00		
Carpool	0.75	1.00		
Bus	1.00	0.75		

A Utility Model of Mode Choice

Example

$U = -T - 5C/Y$

Mode	Time (hours)	Cost ($)	Income y=40	Income y=10
Drive Alone	0.50	2.00	-0.75	-1.50
Carpool	0.75	1.00	-0.88	-1.25
Bus	1.00	0.75	-1.09	-1.38

- Choices depend only on preference ordering
- Nonuniqueness of utility functions

A Utility Model of Mode Choice

Example

U = -T-5C/Y

Mode	Time (hours)	Cost ($)	Income y=40	Income y=10
Drive Alone	0.50	2.00	-0.75	-1.50
Carpool	0.75	1.00	-0.88	-1.25
Bus	1.00	0.75	-1.09	-1.38
Bus	0.75	0.75	-0.84	-1.13

The Need for Random Utility Models

The Need for Random Utility Models

- Inadequacy of Deterministic Utility Models
  - Lack of information for the modeler – about alternatives, about the individual
  - Lack of information for the individual making the choice
  - Similar individuals may make different choices
  - Unexplained variations
- Deterministic Utility Models predict the choices individuals will make with certainty
- Random Utility Models predict the probabilities that each of the available alternatives will be chosen

Limitations of Analysts' Information

- Omission of relevant variables from models
- Measurement errors
- Proxy variables
- Differences between individuals may be ignored
- Day-to-day variations in the choice context may be ignored

Examples of Unexplained Variation in Choice Behavior

Missing variables – example

U_{DA}=-T_{DA}-5C_{DA}/Y + 0.4(A-1)

U_{CP}=-T_{CP}-5C_{CP}/Y

UB = -TB-5CB/Y

Y = 15

Mode	Time (Hours)	Cost (Dollars)	Utility Values		
			0 Cars	1 Car	2 Cars
Drive Alone	0.50	2.00			
Carpool	0.75	1.00			
Bus	1.00	0.75			

Examples of Unexplained Variation in Choice Behavior

Missing variables – example

$U_{DA}=-T_{DA}-5C_{DA}/Y + 0.4(A-1)$

$U_{CP}=-T_{CP}-5C_{CP}/Y$

$U_B = -T_B-5C_B/Y$

$Y = 15$

Mode	Time (Hours)	Cost (Dollars)	Utility Values		
			0 Cars	1 Car	2 Cars
Drive Alone	0.50	2.00	-1.57	-1.17	-0.77
Carpool	0.75	1.00	-1.28	-1.08	-0.88
Bus	1.00	0.75	-1.25	-1.25	-1.25

Examples of Unexplained Variation in Choice Behavior (continued)

- Measurement errors (difference in travel time) – one car

Percent Individuals	20	50	20	10
DA TT (Hours)	0.40	0.50	0.60	0.70
Car Pool TT (Hours)	0.65	0.75	0.85	0.95
U_{DA}	-1.07	-1.17	-1.27	-1.37
U_{CP}	-0.98	-1.08	-1.18	-1.28
U_B	-1.25	-1.25	-1.25	-1.25
Chosen Mode	Carpool	Carpool	Carpool	Bus

Examples of Unexplained Variation in Choice Behavior (continued)

Differences in preferences among individuals

$U^1_{DA} = -0.75\,T_{DA} - 5C_{DA}/Y + 0.4(A-1)$

$U^1_{CP} = -0.75\,T_{CP} - 5C_{CP}/Y + 0.2\{\,A-1)$

$U^1_{B} = -0.75\,T_{B} - 5C_{B}/Y$

$U^2_{DA} = -1.50T_{DA} - 5C_{DA}/Y + 0.4(A-1)$

$U^2_{CP} = -1.50\,T_{CP} - 5C_{CP}/Y + 0.2(A-1)$

$U^2_{B} = -1.50T_{B} - 5C_{B}/Y$

Y=15, same travel time and cost as previous examples

Mode	0 Cars Group 1	0 Cars Group 2	1 Car Group 1	1 Car Group 2	2 Cars Group 1	2 Cars Group 2
Drive Alone			-1.04	-1.42	-0.64	-1.02
Car Pool			-0.90	-1.46	-0.70	-1.26
Bus			-1.00	-1.75	-1.00	-1.75
Chosen Mode						

Multiple sources of unexplained variations in choices

Formulation of the Random Utility Model

U = V+e **'*i***

V = The deterministic component

e = The random component

U_i ? U_j

P(i) = P [$U_i > U_j$ for all *j* in the choice set]

- The dependence of choice on the deterministic component of the utility
- The probability of observing a specific choice

Probabilities in RUM

$$\begin{aligned} P_{ni} &= \text{Prob}(U_{ni} > U_{nj}\ \forall j \neq i) \\ &= \text{Prob}(V_{ni} + \varepsilon_{ni} > V_{nj} + \varepsilon_{nj}\ \forall j \neq i) \\ &= \text{Prob}(\varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj}\ \forall j \neq i). \end{aligned}$$

$$\begin{aligned} P_{ni} &= \text{Prob}(\varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj}\ \forall j \neq i) \\ &= \int_{\varepsilon} I(\varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj}\ \forall j \neq i) f(\varepsilon_n)\, d\varepsilon_n, \end{aligned}$$

where $I(\cdot)$ is the indicator function, equaling 1 when the expression in parentheses is true and 0 otherwise. This is a multidimensional integral over the density of the unobserved portion of utility, $f(\varepsilon_n)$.

$f(\varepsilon_n)$ Is the joint density of the random vector $\varepsilon_n' = \langle \varepsilon_{n1}, \ldots, \varepsilon_{nJ} \rangle$

Various RUM Models

- The linear probability model
- The Logit model
  - Assumes f(.) is Iid extreme value
  - e are uncorrelated over alternatives
  - They have the same variance for all alternatives
- The Probit model
  - Assumes f(.) is a mulitivariate normal
  - With full covariance matrix any pattern of correlation and heteroskedasticity can be accommodated

Various RUM Models

- General Extreme Value (GEV)
  - Generalization of the extreme value distribution
- Mixed Logit
  - f(.) can have any distribution
  - Two part, one contains all the correlation and heterorsedasticity, the other is iid extreme value

The Binary Logit Model

$P_r(1)$ increases monotonically with

$$P_r(1) = \frac{\exp(v_1)}{\exp(v_1) + \exp(v_2)}$$

$P_f(1)$ decreases monotonically with v_z

$$P_r(1) = \frac{1}{1 + \exp[-(v_1 - v_2)]}$$

- **Only differences in utility matter**
- **The scale of utility is arbitrary**

Number of Independent Error Terms

- The probability is a J-dimension integral over the density of the J error terms

$$P_{ni} = \int_{\varepsilon} I(\varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj} \ \forall j \neq i) f(\varepsilon_n) \, d\varepsilon_n.$$

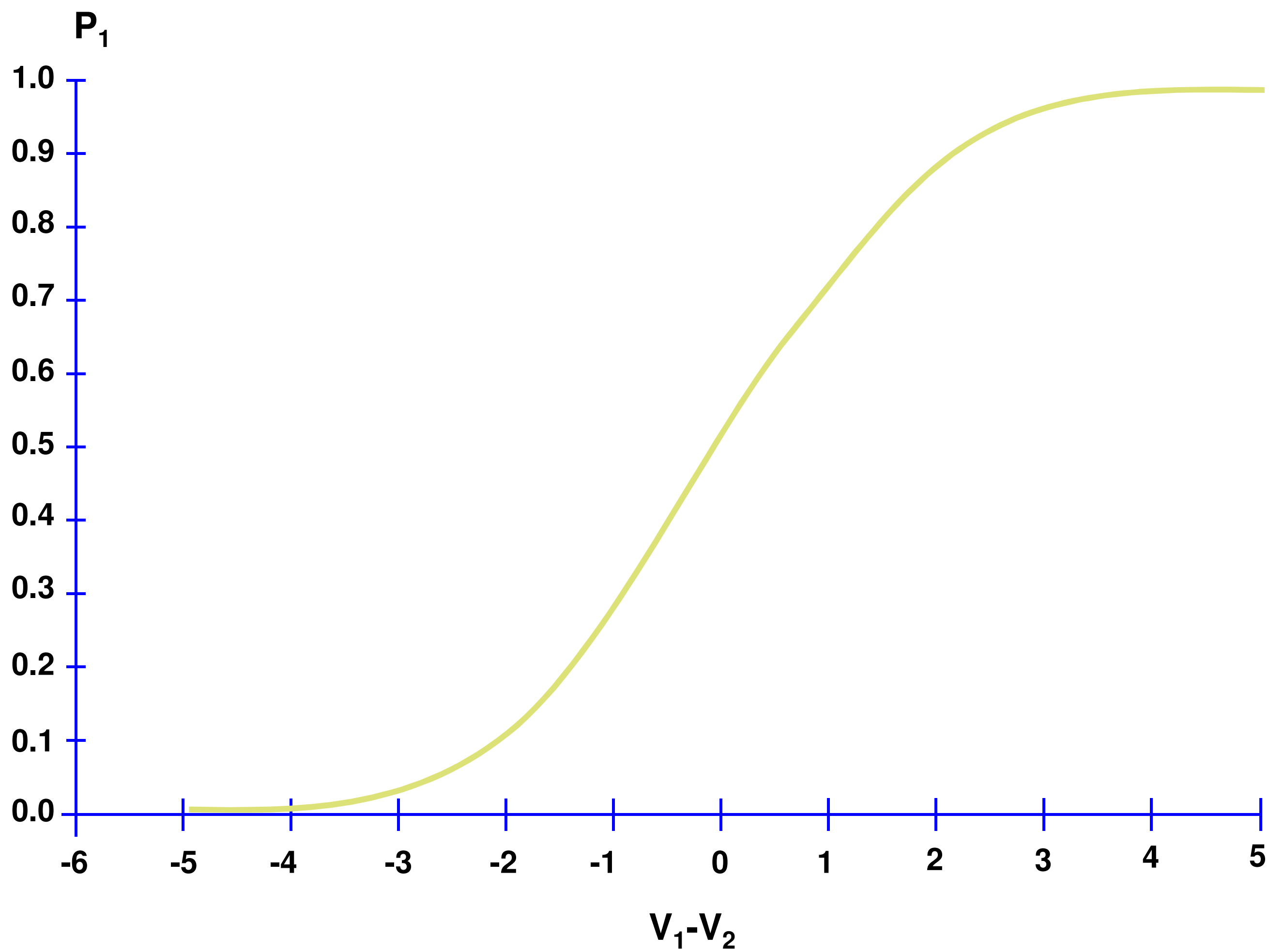
- The dimension can be reduced by recognizing that only differences in utility matter

$$
\begin{aligned}
P_{ni} &= \text{Prob}(U_{ni} > U_{nj} \ \forall j \neq i) \\
&= \text{Prob}(\varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj} \ \forall j \neq i) \\
&= \text{Prob}(\tilde{\varepsilon}_{nji} < V_{ni} - V_{nj} \ \forall j \neq i) \\
&= \int I(\tilde{\varepsilon}_{nji} < V_{ni} - V_{nj} \ \forall j \neq i) g(\tilde{\varepsilon}_{ni})\, d\tilde{\varepsilon}_{ni}
\end{aligned}
$$

where $\tilde{\varepsilon}_{nji} = \varepsilon_{nj} - \varepsilon_{ni}$

$\tilde{\varepsilon}_{ni} = \langle \tilde{\varepsilon}_{n1i}, \ldots, \tilde{\varepsilon}_{nJi} \rangle$ is the $(J-1)$-dimensional vector of error differences, with the . . . over all alternatives except i; and $g(\cdot)$ is the density of these error differences. Expressed in this way, the choice probability is a $(J-1)$-dimensional integral.

The Binomial Logit Model



Specification of the Utility Function

Policy variables/level of service variables

V_a=-T_a-5C_a/y

V_b=-T_b-5C_b/y

Demographic/socioeconomic variables

Example

T_a=0.5hr, T_6=1.0hr

C_a = $1.50, *$C_b$* =$0.50, $1.00

Probability of choosing bus

	C_b = $0.50	C_{to} = $1.00
Low Income (20K)	0.44	0.41
High Income (40K)	0.41	0.39

Low: ΔP_b= -7%

High: ΔP_b= -5%

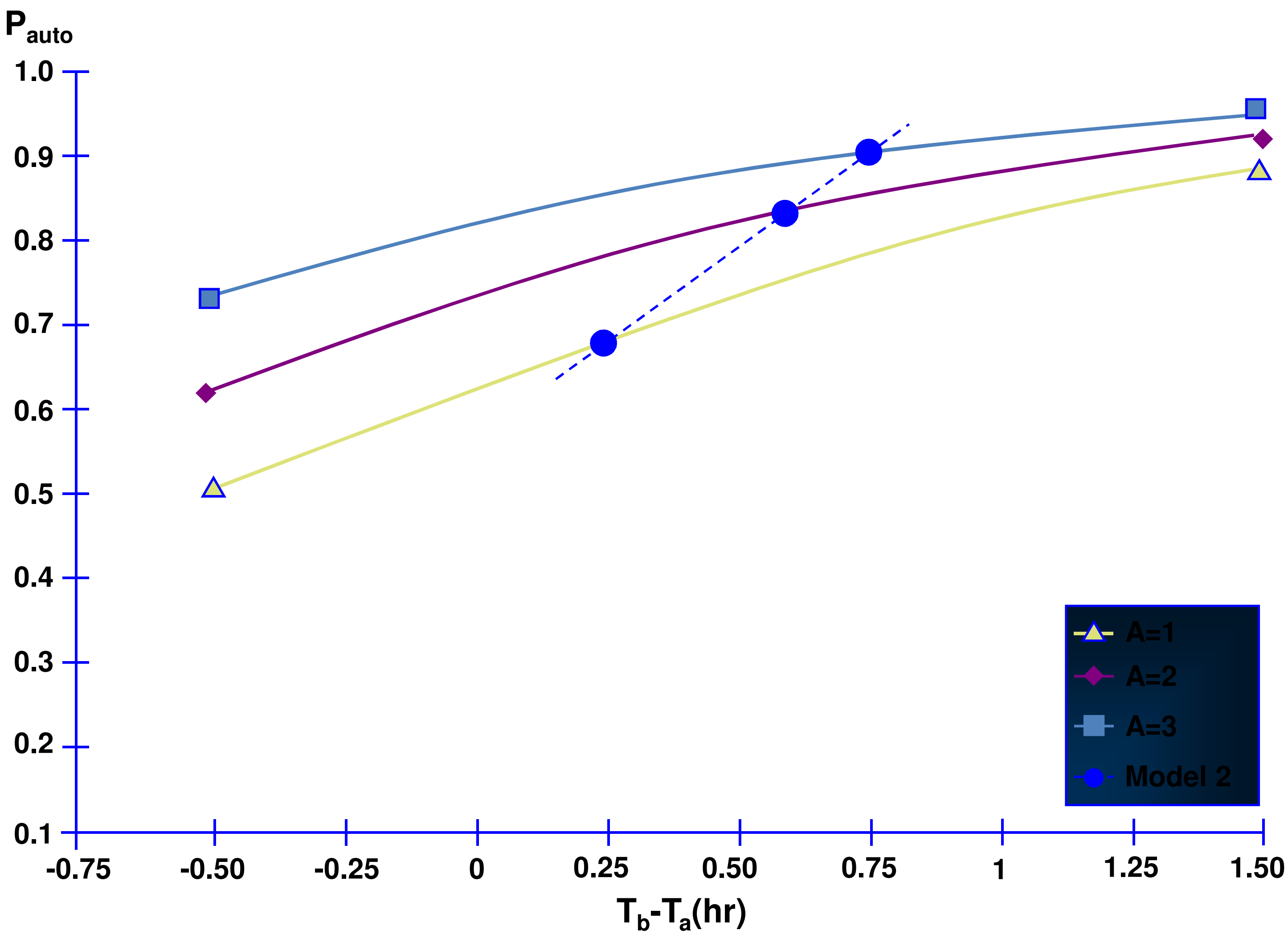
Example – Effects of Omitting a Variable That is Correlated with a Policy Variable

$V_a = -T_a + 0.5\,A$

$V_b = -T_b$

Group	$T_b - T_a$	A	P_{auto}
1	0.25	1	0.68
2	0.60	2	0.83
3	0.75	3	0.90

P_{auto} versus T_b-T_a

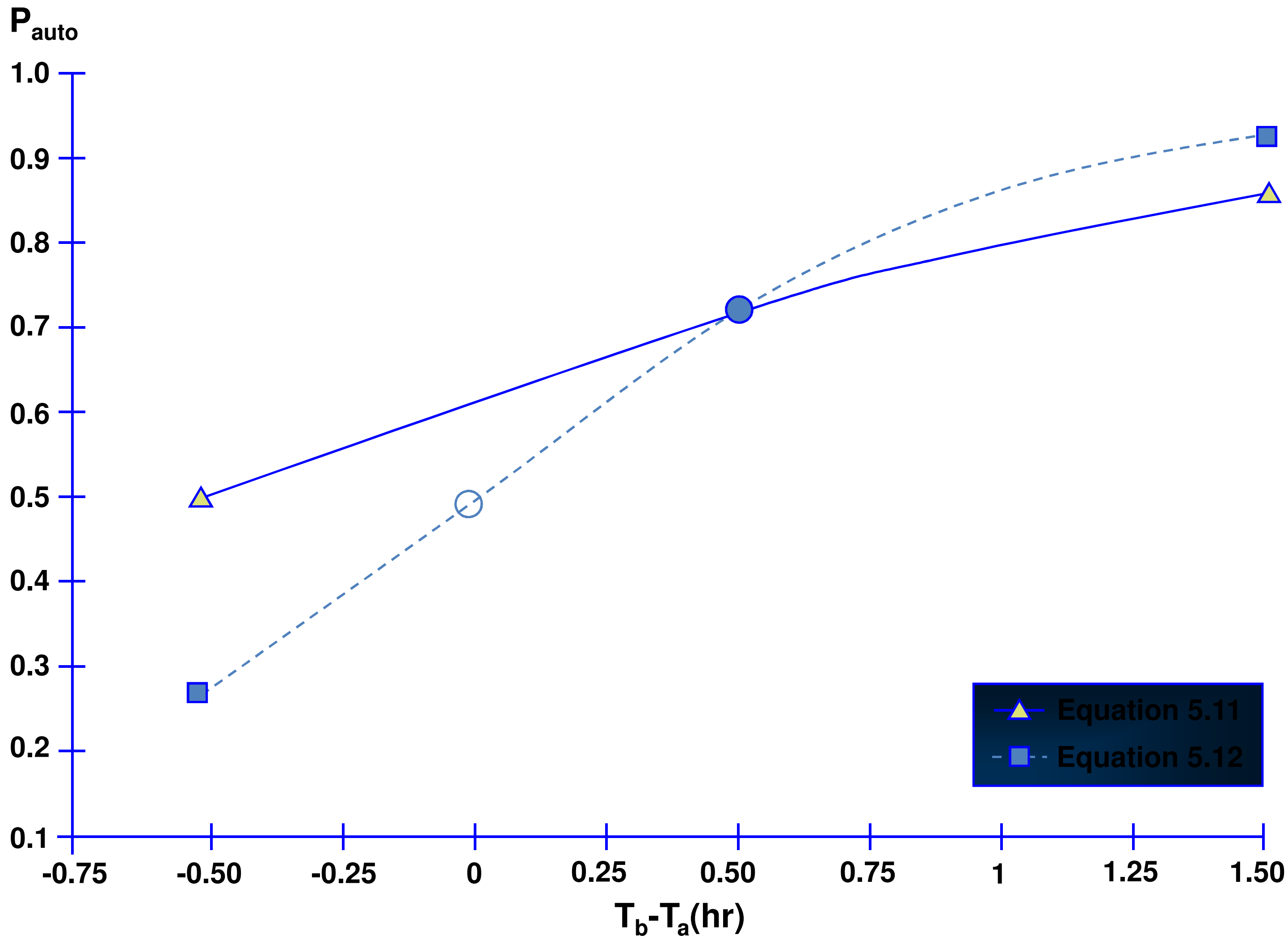


Alternative Specific Constant

$V_a = 0.5 - T_a$

$V_b = -T_a$

P_{auto} versus T_b-T_a



Components of Travel Time

Model 1

$U_A = -T_A$

$U_B = -T_B$

Model 2

$U_A = -0.48TI_A - 1.21TO_A$

$U_B = -0.48TI_B - 1.21TO_8$

$TI_B = 0.5hr$ $TI_A = 0.4hr$

$TO_B = 0.3hr$ $TO_A = 0.05hr$

$T_B = 0.8hr$ $T_A = 0.45hr$

$P^1{}_A = 0.59$ $P^2{}_A = 0.59$

	P_{auto} According to		Change from Base Case	
Case	**Model 1**	**Model 2**	**Model 1**	**Model 2**
Base	0.59	0.59	0.00	0.00
Increase TI_B	0.61	0.60	0.02	0.01
Increase TO_B	0.61	0.62	0.02	0.03

Same effect

Different effect

Generic vs. Mode Specific Variables

Model 1

$UA = -T_A$

$UB = -T_B$

Model 2

$U_A = -4.2T_A$

$U_B = -2.8T_B$

$T_A = 0.45hr$ $T_B = 0.80hr$

$P^1_A = 0.59$ $P^2_A = 0.59$

	P_{auto} According to		Change from Base Case	
Case	Model 1	Model 2	Model 1	Model 2
Base	0.59	0.59	0.00	0.00
Increase T_B	0.61	0.65	0.02	0.06
Decrease T_A	0.61	0.68	0.02	0.09

Same effect

Different effect

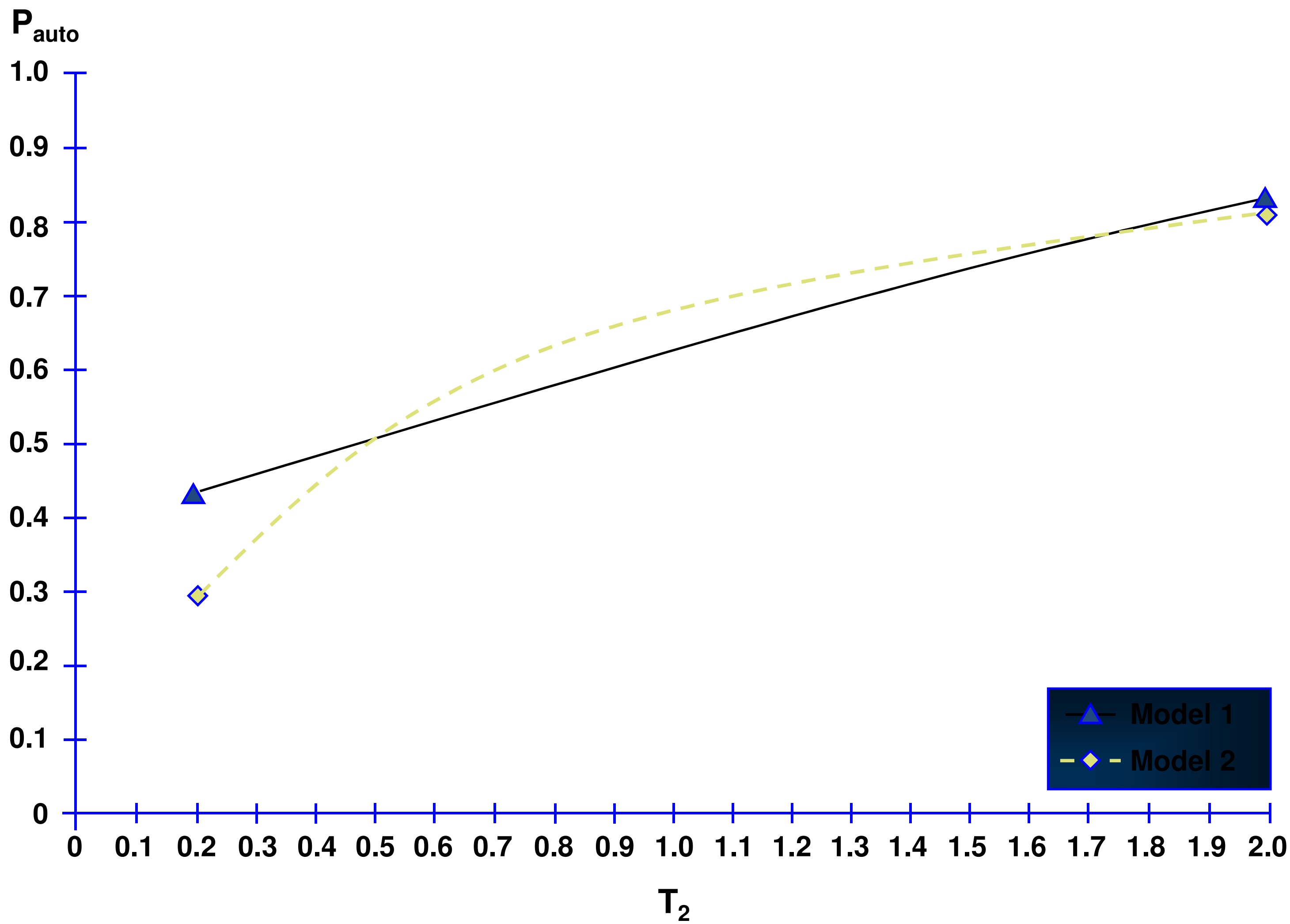
Travel Time

Functional Form

- Travel cost and income
- Auto ownership variables

Travel Time

Functional Form



Estimation of the Logit Model

- Acquisition of data
- Model specification
- Model estimation

The Maximum Likelihood Method

V = a T

Person	Chosen	Travel Time (minutes)	
		Automobile	Bus
1	Auto	50	30
2	Auto	10	20
3	Bus	30	40

Individual 1: P(Auto) = [exp(50a)]/[exp(50a)+exp(30a)]

Individual 2: P(Auto) = [exp(10a)]/[exp(10a)+exp(20a)]

Individual 3: P(Bus) = [exp(40a)]/[exp(30a) + exp(40a)]

Individual 1: P(Auto) = 1 / [1 + exp(-20a)]

Individual 2: P(Auto) = 1 / [1 + exp(10a)]

Individual 3: P(Bus) = 1 / [1 + exp(-10a)]

The Maximum Likelihood Method

(continued)

L = P(Person 1 chooses auto) x P(Person 2 chooses auto) x P(Person 3 chooses bus)

$$\frac{1}{[1+\exp(-20a)]\ [1+\exp(10a)]\ [1+\exp(-10a)]}$$

logL = -(log[1+exp(-20a)] + log[1 + exp(10a)] +log[1+exp(-10a)])