A Stochastic Land Use Equilibrium Model

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Abstract

The problem of static equilibrium in residential location choice is highly complex given its non-linearity and the lack of continuity produced by the set of discrete choices. The problem must take into consideration location externalities associated with the consumers' valuation of neighborhood quality and agglomeration economies, and the assumption of a bid-auction land market. The introduction of the consumers' stochastic behavior has enabled the solution space to be endowed with the property of continuity, which, in addition to the application of efficient fixed-point algorithms, are the foundation of the land use model MUSSA (Land Use Model of Santiago). In an effort to complement and enrich this approach, this paper extends the assumption of stochastic behavior to the real estate market, the supply side, including production economies of scope and scale, generating a system of fixed-point equations under a common platform based on the Gumbel distribution. This logit formulation of the system and the analysis of the fixed-point algorithm are described in this paper. Simulations demonstrate the stability of solutions and their convergence to well-behaved unique solutions independent of the starting point.

Keywords

Land use, equilibrium, logit, International Conference on Travel Behaviour Research, IATBR

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1. INTRODUCTION

The accumulated research in the study of urban land use demonstrates the system's high degree of complexity when all the interactions amongst consumers and suppliers are included, generating a complex non-linear mathematical problem. The complexity is due to the diversity of agents with different but interdependent behavior. Households, for example, can be differentiated by socioeconomic attributes, whereas businesses/commercial activities can be differentiated by economic activity and business size. Additionally, real estate options provided by suppliers are all differentiated goods, distinguished by attributes of construction as well as location. Finally, the State plays a strong role by imposing a variety of regulations that affect the market.

In previous research, (Martínez, 2000), this problem has been theoretically specified using the following combination of assumptions:

- Urban land is a quasi-unique good bought and sold in an auction-type market.
- Real estate options are discrete and differentiated units defined by the location zone and building type.
- The available location is assigned to the best bidder.
- Bids are made by households and firms that compete for real estate options.
- Consumers' bids are assumed to be random variables, thus considering the idiosyncratic nature of the agents.
- Consumers' bids are assumed mutually interdependent as their location choices define the neighborhood quality for residences and the agglomeration economies for non-residents.
- The supply side is represented by a time-series deterministic model that predicts the the number of supply units by zone and building type based on price.
- Equilibrium is achieved when all consumers find a location somewhere in a static framework.

These assumptions describe an imperfect competitive market, where the prices are defined endogenously by an auction mechanism as consumers simultaneously find a location through the rule of the best bidder. The explicit modeling of location externalities, namely neighborhood quality and agglomeration economies, generates equilibrium conditions described by non-linear mathematical problems. This model was called the Random Bidding Model (RBM) and has been developed into operational software using a specific solution algorithm, called MUSSA, currently applied to the city of Santiago (Martínez and Donoso 2001).

This paper develops a new model of equilibrium, which is an extension of the RBM model. The principal modification is in the adoption of new assumptions about how supply agents behave in the competitive market, in this case by adding idiosyncratic variability. This variability is modeled by introducing randomness in the in the supply side within the approach of static equilibrium. A static maximum profit logit model replaces the times-series sub-model of deterministic supply in MUSSA , which requires the study of some important issues . The first topic is the indetermination of absolute values for prices and rents predicted by the new model. Indeed, the probabilistic supply model under depends solely on relative price values such that absolute values cannot be identified generating the price indetermination. The second topic is the change in mathematical form of the equilibrium model, which requires the study and development of a specific solution algorithm. Furthermore, in this paper we have added an improvement to the consumers' behavior, which consists of explicitly including the income restriction in agents' behavior.

Our new model is called Random Bidding and Supply Model (RB&SM). It has the advantage of offering better consistency in the behavior of all the agents in the system and it also takes better advantage of the mathematical structure generated by logit models. Most important is the fact that the new supply model can incorporate the effects of economies of scale and scope in the real estate market, which are prominent responsible for densification and sprawl tendencies.

Following the proposal of the model in section 2, the solution algorithm and an analysis of performance are presented in section 3.

2. MODEL FORMULATION

2.1. Assumptions

The urban land market assumes imperfect competition in that a real estate is defined by unique combinations of building and location attributes, giving it a differentiated character. This unique character arises from the fact that urban location is valued for what surrounds it (neighborhood, parks, infrastructure, etc.) which cannot be reproduced by a production process. For this reason, the market behaves like an auction, which is taken into account in rent or urban economic theory developed initially by Alonso (1964). Under this condition, goods go to the highest bidder, where the bids represent consumers' willingness to pay (WP). Thus, consumer behavior is modeled by the WP function instead of by utility. This valuation function includes attributes that describe the complex interaction amongst consumers, called location externalities, which constitutes another argument for market imperfection, and at the same time introduces great analytical complexity to the model.

Furthermore, an extension of the Random Utility Theory is applied, which was developed for the case of auctions (Martínez, 1992), and profit maximization (Anas, 1982). In practice this implies that both the consumers (who want to choose a location) and the suppliers will be modeled under the assumption that the function that describes their behavior – WP and profit respectively – follow a random distribution, therefore decisions are represented by a probability. Additionally, in this model the set of location options are discrete and defined by zone and building type. This finite set of alternatives constitutes the discrete space where those who supply and those who demand make their decisions. In this way, the model also falls inside the scope of the Discrete Choice Theory.

Although the model can be defined as dis-aggregated at the level of each agent and location, in practice aggregated versions are used such as the one presented here. Consumers are classified into socio-economically homogeneous categories (index h) and the supply described by location or zone (index i) and property type (index v).

2.2. Demand Model

The variable that describes the consumer behavior is the bid of each agent, represented by the WP for the location to achieve a certain level of utility. The theory that relates both functions, utility and WP, shows that WP represents an expenditure function (in all goods except location) associated with the problem of maximization of consumer utility and it is derived inverting– in the property price – the indirect utility function conditional on the location (Solow 1973; Rosen 1974; Martínez 1992).

This theory yields the following relationship between the indirect utility function V and willingness to pay, hereafter called bids: $B_{hvi} = I_h - V_h^{-1}(z_{vi}, P, U_h, \beta_h)$, with I_h the household income, P the price vector for goods and β_h the utility taste parameters; z_{vi} is the vector of real estate attributes. It is possible to demonstrate that for the bids thus defined, the location where the agent is the highest bidder is that of the maximum surplus or maximum utility (Martínez, 1992, 2000).

In the RB&SM model we assume I_h, P and β_h as constant parameters (hence hidden in what follows), plus the following additive condition: $V_h^{-1}(z_{vi}, P, U_h) = I_h - f^1(U_h) - f^2(z_{vi})$. This additive assumption imposes relevant constraints into the model, but it also introduces significant benefits in calculating the equilibrium, allowing the model to deal with complex non-linearities like the explicit modeling of complex location externalities. The assumption yields:

$$B_{hvi} = b_h^1 + b_{hvi}^2(P_{\bullet/\bullet i}, S_{\bullet i}) + b^3 \tag{1}$$

where the bid components are:

- b_h^1: adjusts utility levels to attain equilibrium.
- b_{hvi}^2: describes the valuation of property attributes. Some attributes are exogenous to the location and land use distribution, like rivers, parks, hills, etc., then they are represented by zone attractive parameters in this term. The most complex attributes are those endogenous, which describe location externalities and are defined by two types of variables. First

the (probability) distribution of agents in the zone, given by $P_{\bullet/\bullet i}$ [1] that describes attributes like neighborhood quality by combining the characteristics of agents located in the zone (all building types) with the number of agents there located. Second, the building stock supplied in the zone (all buildings), $\overline{S}_{\bullet i}$, which describes the building environment in the zone.

b^3: is a term independent of consumers and supply options, which adjusts bids to absolute levels in the whole market. This component is most relevant in the calculation of absolute value of rents and for the adjustment of total supply to changes in the total demand, as shown below.

In the case of firms (non-residential activities), their WP function is derived from the profit function for each economic sector or industry, assuming intra sector behavior homogeneity. In this case, it is also assumed that the bid function is additive, like in (1).

In order to include the behavior variability produced by idiosyncratic differences between consumers within a cluster, bids are assumed to be a random variables: $\widetilde{B}_{hvi} = B_{hvi} + \varepsilon_{hvi}$, with random terms ε_{hvi} distributed Gumbel, identical and independent (IID). The Gumbel distribution is justified by Ellikson (1981) noting that it is consistent with the maximum bidding process of the auction, where only the maximum bid within a cluster is relevant for the auction. This assumption has also important practical consequences in the solution algorithm. From these assumptions, the (multinomial) probability that one of the $\overline{H}_h$ [2] agents type h is the highest bidder in (v,i), is yield -conditional on the supply being available- by:

$$P_{h/vi} = \frac{\overline{H}_h \exp(\mu B_{hvi})}{\sum_g \overline{H}_g \exp(\mu B_{gvi})} \tag{2}$$

[1] Notation: $x_{\bullet k}$ denotes the vector of all elements of x whose second component is k.

[2] Overlined variables denote exogenous information required by the model.

where the parameter μ is inversely proportional to the variance of the bids. The parameter $\overline{H}_g$ is the number of agents type g who participate in the market. Here the aggregated version of the multinomial logit probability is utilized which includes the correction for different sizes between agents clusters, as proposed by McFadden (1978).

Future versions of this model may drop the IID assumptions to allow for different degrees of correlation between random terms, which can be developed based on available research on discrete demand model, however further research will be required to generate the suitable equilibrium algorithm for each alternative model specification.

Thus the demand model is:

$$P_{h/vi} = \frac{\overline{H}_h \exp(\mu(b_h^1 + b_{hvi}^2(P_{\bullet/\bullet i}, S_{\bullet i})))}{\sum_g \overline{H}_g \exp(\mu(b_g^1 + b_{gvi}^2(P_{\bullet/\bullet i}, S_{\bullet i})))} \tag{3}$$

where b^3 is cancelled out. In a synthetic form this is written as:

$$P_{h/vi} = P_{h/vi}(b_\bullet^1, P_{\bullet/\bullet i}, S_{\bullet i}) \quad \forall h, v, i \tag{4}$$

This equation represents the location fixed point, with the probability variable both in the right and left hands of an unsolvable equation. It mathematically describes the interdependence between consumer decisions, i.e. location externalities, in which the location of an agent depends on locations of other agents (households and firms) in the same zone.

As a result of the auction, the rent of a real estate, identified by type v and zone i, is determined by the expected value of the highest bid, which thanks to the Gumbel distribution it is the known logsum or implicit value function, given by:

$$r_{vi} = \frac{1}{\mu} * \ln\left(\sum_{g} \overline{H}_g \exp(\mu B_{gvi}) \right) + \frac{\gamma}{\mu} \tag{5}$$

that can be decomposed in two for the benefit of a later exposition:

$$r_{vi} = \frac{1}{\mu} \ln\left(\sum_{g \in H} \overline{H}_g \exp\left(\mu\left(b_g^1 + b_{gvi}^2\right)\right) \right) + b^3 + \frac{\gamma}{\mu} = \bar{r}_{vi} + b^3 \tag{6}$$

Then the rent depends on bids B_{hvi} and these in turn on the all other variables. It is worth bearing in mind that, despite the elegance of this equation, it says that rents are proportional to the total number of consumers, which is clear if we consider the case where all agents have identical bids $B_{vi,}$ yielding $r_{vi} = B_{vi} + \frac{1}{\mu} \ln\left(\sum_{g \in H} \overline{H}_g \right)$; this might not be always an desired property for rent as total population increases, so we recommend careful analysis in actual applications.

2.3. Supply Model

The behavior of the real estate suppliers consists of deciding what combination of building and zone (v,i) would generate the maximum profit, subject to prevailing market regulations. The profit function, denoted as π, is calculated as the difference between the rent (r_{vi}) that will be obtained for a supply option and the cost incurred (c_{vi}), including land, construction and maintenance costs. We define π ´s as profit per unit of production, then the total profit is:

$$\pi_{vi} = S_{vi}(r_{vi} - c_{vi})$$

There are some theoretical aspects to define for the design of the supply model. One is the assumption of homogeneity of profits with respect to market conditions of information and mobility of resources. It is worth noting that urban markets are highly regulated by zoning regulations, defined both by zone and building type, hence it is plausible that profit may be different by sub-markets defined by (v,i).

A second important aspect is the heterogeneity of the suppliers, which occurs when suppliers have different profit functions. This function may be different depending on various sources of heterogeneity, for example the size of the firm that may imply different access to technology and generate different fixed costs. Then the model should permit different profit functions by types of developer (clusters), thus $c_{vi} = (c_{vij}, j \in (1, J))$, J the number if developers' clusters.

Another theoretical aspect is the level of profit aggregation that the supplier maximizes. In the presence of scale economies (intra sub-market economies), denoted as $c_{vij} = c_j(S_{vij})$, each supplier j should define the optimal production level S_{vij} by maximizing profit in each sub-market independently. Alternatively, in the presence of economies of scope (inter sub-markets economies), the rational behavior must consider the use a more complex strategy looking for an optimum combination set of supply options in all sub-markets. The more general case including full interdependency reflected in costs functions denoted as $c_{vij} = c_j(S_{\bullet\bullet})$, where production cost depends on what is built everywhere by every builder. Less complex interdependencies are of course likely to occur in real markets. In any case, the supplier must determine the optimum amount to produce in each sub-market (v,i), that means to define an optimal vector $(S_{..j})$.

Then, the general problem of the j^{th} supplier may be formulated including economies of scale and scope, written as:

$$\begin{aligned} & \underset{S_{..j}}{Max}\ \pi_j = \sum_{vi} S_{vij}\pi_j(r_{vi}(S_{\bullet\bullet\bullet}) - c_j(S_{\bullet\bullet\bullet})) \\ & s.a. \quad S_{vij} \in (R_i, T_{vij}) \quad \forall v, i, j \end{aligned} \tag{7}$$

The set of restrictions indicates that supply must comply with the set of regulations at each zone *(R_i)* and is constrained to the technology available to the building sector (T_{vij}).

In order to develop an operational supply model, notice that in the context of the RB&SM the suppliers' problem (7) rents are random variables, hence profits are also random. Moreover, by the property of conservation of the Gumbel distribution under maximization, rents are random variables with a Gumbel distribution that preserves the same scale parameter μ of the bids functions defined above. Thus, in a model with deterministic costs profits would be Gumbel distributed IID with the same scale factor as the bids.

The RB&SM makes the following assumptions to obtain an operational supply model. We assume the developers' profits as independent, identical and Gumbel distributed variables with scale parameter λ (different to the demand model parameter). Thus, the expected number of residential supply units type (v,i), S_{vi}, is given by the multinomial probability that this unit type is the maximum profit option for each developer in the industry times the developer's share of the market. This is:

$$S_{vi} = S\sum_{j}\rho_j \frac{\exp(\lambda(r_{vi} - c_{vij}))}{\sum_{v'i'}\exp(\lambda(r_{v'i'} - c_{v'i'j}))} \tag{8}$$

where λ is inversely proportional to the profit variance and S is the total supply in the urban area. ρ_j is the developer's j share of the market. The multinomial logit is the conditional probability, given the developer, to produce a real estate unit in sub-market (v,i).

It is worth commenting the supply model (8). It represents a more limited model than the general formulation in (7), as it assumes the following additive condition: $Max\ \pi_j = \sum_{vi} S_{vij} Max\ \pi_{vij}$, which holds only if individual profits π_{vij} are independent across real state sub-markets. Such condition hardly holds if the technology of the building industry generates economies of scope, because in this case building costs, denoted as $c_{vij} = c_j(S_{\bullet\bullet})$, are explicitly dependent on the mixture of production in each sub-market in the production set of the developer. Nevertheless, it is theoretically possible to model such dependency – completely or at least partially- by the cost function itself, leaving no correlation to the random terms which are then yield independent. In the applied field, however, it would be wise to explore more complex logit model structures than the multinomial, which will remain for future research.

From the rent equation (6), it can be seen that b^3 is cancelled out in equation (7), then the reduced form of the supply model is:

$$S_{vi} = S \cdot P_{vi}(b_{\bullet}^{1}, P_{\bullet/\bullet\bullet}, S_{\bullet\bullet}) \tag{9}$$

which represents the fixed point equation of the non-linear supply model.

It is important to note that this model does not yet include the constraints imposed in (7) to represent zoning regulations (R). Technology restrictions (T) may only included in the profit specification in the functional form of costs.

Modeling zoning regulations is a fundamental feature of a land use model, specially to make it applicable as a design tool for zoning plans. In the RB&SM model, we incorporate linear regulations of the following form: $\sum_v a_{vi}^k S_{vi} \leq R_i^k$, where the coefficients a^k –associated to the k^{th} restriction- and the restrictions values R_i are all exogenous parameters of the model. Of course the linear form limits the diversity of regulations that can be considered, but they are sufficient for the great majority of actual urban regulations.

In order to incorporate linear regulations in the model, we build upon a bulk of research on these type of problems, specially that on entropy models. See for example the model proposed recently by Martínez and Roy (2003). We define a "restricted profit" function, which is the feasible profit given the set K of regulations in the city, with K_i regulations in each zone *i,* which is $\pi_{vij} = r_{vi} - c_{vij} - \sum_{k \in K} \gamma_i^k$. The set of γ parameters are known as "balancing factors" that adjust supply to zone restrictions at each zone.

Nevertheless, we know that $\gamma_i^k \geq 0 \; \forall i,k$, because they represent the lagrangian multiplier for each constraint and represent the increase in profit associated with a relaxation of the respective regulation. The direct effect of a positive parameter is that expected profits is reduced and so is supply, however there are second order effects potentially in the inverse direction produced by the non-linear role of γ parameters in rents and costs.

We know that only one constraint (denoted by $\bar{k}$) is actually binding at each zone, whose corresponding parameter is denoted by γ_i and called the "binding parameter". Then, observe that:

- $\gamma_i^k = 0, \forall (k \neq \bar{k} \in K_i); \gamma_i^{\bar{k}} = \gamma_i \geq 0$; and $\sum_i \gamma_i = \sum_{k_i \in K} \gamma_i^{k_i}$.
- $\gamma_i = max_{k \in K_i} \; \gamma_i^k$; the binding parameter is the superior of all parameters in each zone.
- $\sum_v a_{vi}^k S_{vi}(\gamma_i) < R_i^k \; \forall k \neq \bar{k}$, i.e. the binding parameter γ_i assures that all not binding constraints hold, so their respective parameters can be assumed equal to zero.

An important implication is that the number of parameters needed to be calculated is not equal to the number of constraints, but to the much smaller number of zones; however, an efficient algorithm will be required to identify the binding regulation without calculating all γ parameters.

The set of binding parameters may be calculated with the following formula:

$$\begin{aligned} \gamma_i &= \frac{1}{\lambda} \ln \left[\frac{S}{R_i^k} \sum_{vj} a_{vi}^k \rho_j \exp(\lambda(r_{vi} - c_{vij} - \tilde{\pi}_j)) \right] \\ \text{with} \quad \tilde{\pi}_j &= \frac{1}{\lambda} \ln \left[\sum_{v'i'} \exp(\lambda(r_{v'i'} - c_{v'i'j} - \gamma_{i'}) \right] \end{aligned} \tag{10}$$

where we have replaced equation (8) and the restricted profit into the binding linear constraint, then we solved for γ_i . Equation (10) represents another fixed point problem, because each γ_i is a function of itself and all other elements of the γ vector; additionally, the parameters' logsum function can not be analytically solved for each γ_i .

These parameters have a relevant practical economic interpretation: they represent the marginal profit obtained by suppliers –by the industry, not by individual supplier- if the corresponding regulation is marginally relaxed, usually called the shadow prices. This price is in fact an increase in the production cost of real estate supply, understood as an increase in land lot prices, hence it represents a capitalization on land prices of a monopoly power generated the regulation. They can be used as an index to asses each regulation by the impact in the economy. Additionally, $\tilde{\pi}_j$ is interpreted as the expected constrained profit for the j^{th} developer under the regulated market.

2.4. Equilibrium

Here we study the supply-demand auction-Walrasian equilibrium. There are several alternative specifications of equilibrium for the urban-land use market, from that defined by every agent is located somewhere, i.e. demand is thus satisfied but supply may not be fully used, to a more demanding version in which demand and supply are equal. All these options are represented by:

$$\sum_{vi} S_{vi} P_{h/vi} \geq \overline{H}_h \qquad \forall h \tag{11}$$

in which equilibrium is verified for each consumer category h and for all of them simultaneously. In this paper, we analyze the equality case, which represents the classical static equilibrium. The inequality condition, also referred to as dis-equilibrium, leads to dynamic formulations that involve a number of other considerations beyond the scope of this paper.

The equality condition is met if b^1 verifies that:

$$b_h^1 = -\frac{1}{\mu} \ln\left(\frac{1}{\overline{H}_h} \sum_{vi} S_{vi} \exp(\mu(b_{hvi}^2 - \bar{r}_{vi})) \right) \tag{12}$$

this equation is obtained solving (11) for b_h^1 under equality. As r_{vi} depends on the bids in (6), this equation can be written in a reduced form as:

$$b_h^1 = b_h(b_{\bullet}^1, P_{\bullet/\bullet\bullet}, S_{\bullet\bullet}) \tag{13}$$

and constitutes another fixed point, this time in vector b^1 whose solution verifies equilibrium conditions. This problem has the same logsum functional form as the $\gamma's$ equation (10), because the equilibrium condition also represents a set of linear constraint.

Now it is time to comment on the previous interpretation of the term b^1 (section 2.2) as the model variable that “adjusts utility levels to attain equilibrium”. The additive assumption on bids yields this variable with the following interpretation: it represents a monetary equivalent of clusters’ utility levels, with b^1 negatively related with utility: the higher the bid for a location the lower the utility obtained (all location attributes held constant). Then the values obtained from (12) represent an index of the utilities attained by agents’ clusters at equilibrium. As expected, ceteris paribus and neglecting second order effects caused by non-linearities, this index increases with $\overline{H}_h$, so utility decreases with population because supply is more demanded; more supply increases utility while higher rents have the opposite effect.

2.5. Additional restriction by rents and income

The previous model is based on the specification of a WP function or a bid that should meet the restriction of household income and, if possible, reproduce absolute values of rents and bids. Here we propose methods to attain these restrictions.

The values of rents previously defined are relative within the model until the term b^3 is identified. One method is to assume that total supply depends on absolute values of rents and external macroeconomic variables (X), then

$$S = S(r, X), \quad r = (r_{vi}, \forall v, i) \tag{14}$$

where r may be, for example, the average rent or the maximum rent across the city. Since in static equilibrium S is known and equal to the total population of agents, (14) can be solved for b^3. An alternative method is to define a relationship between absolute rents and the external macroeconomic variables, for example with reference to the rents of a given zone m, r_m; usually m is a zone at the city limit where the land value is related to agricultural land values. Then:

$$r_m = f\left(X_m\right) = \bar{r}_m + b^3 \tag{15}$$

with $\bar{r}_m$ given by evaluating the rent model (6) for location m, which yields b^3 directly. The best approach is that which has the best empirical support. From either method we have an expression given by:

$$b^3 = b^3(b^1_{\bullet}, P_{\bullet/\bullet\bullet}, S_{\bullet\bullet}) \tag{16}$$

that allows the calculation absolute values of rents and bids. It is important to emphasize that from the point of view of the model, b^3 does not alter the equilibrium solution in (b^1,$P_{\cdot/\cdot\cdot}$,$S_{\cdot\cdot}$), it only affects the absolute values of the rents and bids. But of course this occurs principally due to the functional forms of the bids and profits used in the RB&SM model.

Additionally, the model must also meet the agents' income restrictions, which are:

$$(Px)_{hvi} + r_{vi} \leq I_h \quad , \quad \forall h \text{ located at } (v,i) \tag{17}$$

where x is the vector of consumers goods, which we assume continuous, and P its corresponding price vector. If we also assume good enough information, the auction is capable of extracting the maximum value possible from consumers' WP, then equation (17) must hold for equality:

$$(Px)_{hvi} = I_h - \bar{r}_{vi} + b^3 \tag{18}$$

This allows to estimate the level of consumption of goods differentiated by location and cluster, which constitutes interesting information that directly and explicitly links location and consumption. We note that, from (18), a differential rent between two locations induces a differential expenditure in goods that is exactly compensated –on expected not necessarily actual values- by an equivalent differential in the utility associated with the respective location amenities (attributes).

3. The Equilibrium Solution

In this section we analyze how to solve the problem of static equilibrium, where we propose an algorithm and analyze its properties.

3.1. System of Equations

The static equilibrium of the urban land use system is represented by the simultaneous solution of the previous set of equations, that together can be written like a macro fixed point problem such as:

$$\begin{aligned}
&P_{h/vi} = P_{h/vi}(b^1_{\bullet}, P_{\bullet/\bullet i}, S_{\bullet i}, \gamma_i) && \forall h, v, i \\
&S_{vi} = S \cdot P_{vi}(b^1_{\bullet}, P_{\bullet/\bullet\bullet}, S_{\bullet\bullet}, \gamma_{\bullet}) && \forall v, i \\
&b^1_h = b_h(b^1_{\bullet}, P_{\bullet/\bullet\bullet}, S_{\bullet\bullet}, \gamma_{\bullet}) && \forall h \\
&\gamma_i = \gamma_i(b^1_{\bullet}, P_{\bullet/\bullet\bullet}, S_{\bullet\bullet}, \gamma_{\bullet}) && \forall i
\end{aligned} \tag{19}$$

which is a system of dependent non-linear equations with dimension [(#h+1)(#v#i)+(#h)+(#i)], with the same number of unknown variables. This system is complemented with the equation for the absolute rents:

$$b^3 = b^3(b^1_{\bullet}, P_{\bullet/\bullet\bullet}, S_{\bullet\bullet}, \gamma_{\bullet}) \tag{20}$$

that does not intervene in the solution of system (19). The solution vector is $(b^1*, P*, S*, \gamma*, b^3*)$ from which we can calculate the bids, location patterns, rents and profits.

3.2. Solution Algorithm

We now focus on the algorithm to solve this equation system and the analysis of the solution sensitivity to model parameters. It should be noted that in spite of the assumptions taken, this is a highly complex non-linear system of equations and therefore there are no general solution tools. It is known that the form that is given to the functions involved is very important in the behavior of any solution algorithm, in our case this affects the functional form assumed for bids and production costs in each specific application of the model. It is also known from experience that in large complex problems, the most efficient and robust algorithms are those that take advantage of the structure of the equations involved. In our model, this is highly important because all the equations are derived from the Gumbel distribution, which defines a unified platform for the mathematical problem. Indeed, the first two equation in (19) are multinomial logit formulae, while the last two are logsum formulae. Therefore, the algorithm is only valid for the above specification of the RB&SM model.

The main solution algorithm is the following:

Define the generic vector

$$x = (x_j, j \in (1,2,3,4); x_1 = (b_h^1, \forall h), x_2 = (P_{h/vi}, \forall hvi), x_3 = (S_{vi}, \forall vi), x_4 = (\gamma_i, \forall i))$$

Call Check, verifies for feasible regulation set

Initialize: n=0, m=0

Iterate the equation system

4.1 $n=n+1$, $t=1$, $j=1$

4.2 if $j=4$, (if m=0, $x_4=0$, call Binding) (m=0 unrestricted, m=1 calls binding)

4.3 $x^t = (x_j(x_j^{t-1}, x_k = \bar{x}_k \ \forall k \neq j), \ \bar{x}_k \ \forall k \neq j)$

4.3 if $\left| x_{jl}^t - x_{jl}^{t-1} \right| > e_j \ \forall l$, $t=t+1$, go to 4.2 (go to recalculate x^t)

4.4 $\Delta x_j = x_j^t - \bar{x}_j, \quad \bar{x}_j = x_j^t$

4.5 if $j<4$, ($t=1$, $j=j+1$, go to 4.2) (go to adjust new variable x_j)

Global convergence

5.1 If $\Delta x = (| x_j(\bar{x}_j, \forall j) - \bar{x}_j | > e_j, \forall j)$, go to 4.1 (iteration n+1 of fixed points)

5.2 If m=1 stop, $x^* = (\bar{x}_j, \forall j)$, (n=0, m=m+1 go to 4.1).

The Check procedure:

Find an initial S: If a is invertible $S^0 = R \cdot a^{-1}$.

Adjust for non-negativity:

$$V_i = (v / S_{vi}^0 < 0), S^0 = (S_{vi}^0 = 0 \ \forall v \in V_i, S_{vi}^0 = S_{vi}^0 + \Delta S_{vi})$$

$$\text{where} \sum_{v' \notin V_i} S_{v'i}^0 a_{v'i}^k = \sum_{v' \notin V_i} (S_{v'i}^0 + \Delta S_{v'i})' {}^* a_{v'i}^k$$

if $\exists! \, S_{vi}^0 < 0$ or $\bar{S} = \sum_{vi} S_{vi}^o > \sum_h \overline{H}_h$, then the regulation set is not feasible else $x_3^0 = S^0$.

The Binding procedure:

i) Starting values: $\gamma^0 = \{\gamma_i^k = 0, \forall i, k\}$

ii) Constraints evaluation:

$$\Delta_i = \min_{k \in K_i}\left(R_i^k - \sum_{v'} (a_{v'i}^k S_{v'i}) \right) \; ; \; k_i = \arg\min_{k \in K_i}\left(R_i^k - \sum_{v'} (a_{v'i}^k S_{v'i}) \right) \; ; \; \bar{\gamma}_i = (\gamma_{k_i}, \forall i)$$

Iterations n=1

iv) if $\Delta_i \leq 0$, $\bar{\gamma}_i^n = \gamma_i\left(\bar{\gamma}^{n-1}\right) \quad \forall i$

if $\Delta_i \geq 0$, $\bar{\gamma}_i^n = 0 \quad \forall i$

v) if $\left|\gamma_i^n - \gamma_i^{n-1}\right| > e \;\; \forall i$, n=n+1 go to iv)

Final parameter: $\gamma^* = \{\gamma_i^k = \gamma_i^n, \forall i, k\}$

The algorithm is simple, it sequentially solves for each of the four variable vector x_j the corresponding fixed point equation until convergence is attained, called a local t-iteration; then the next vector x_{j+1} is adjusted and so on until the whole vector is adjusted, which completes a general iteration. Within a local iteration, the fixed point is solved for all elements of the vector $(x_{ji} \in x_j, \forall i)$ simultaneously, by simply repeatedly evaluating the variable in the corresponding fixed point function, holding the other variables fixed x_{-j} at their current values. Some times this is called the picking algorithm, which in this case is applied to a set of fixed points. The local iteration procedure converges once the variables are within a tolerance value; global convergence requires convergence in all variables.

The exceptions to this general procedure are associated with the presence of zone regulations. First, the algorithm starts (line 2) with the Check procedure that verifies if the feasible supply space defined by the regulation set is feasible (line 2), thus the procedure finds a point $S_{vi}^0 \geq 0, \forall v, i$ such that $\sum_{vi} S_{vi}^0 \geq \sum_h \overline{H}_h$ and $\sum_v a_{vi}^k S_{vi}^0 \leq R_i^k$, $\forall i$. Second, the equilibrium is first solved ignoring all regulation constraints (m=0) to find the unrestricted solution; this solution is taken as the starting point to apply the algorithm for the constrained problem (m=1). This procedure avoids getting an ill solution at the age of the feasible space when there is a solution in the interior (not binding) of that space; indeed, an starting point for the restricted iteration (m=1) out of the feasible region, always gives a solution at the edge of that region, never in the interior. It is worth noting that once the algorithm starts from a point in the interior of the of the feasible space, the solution global solution is unique (except when the logit model tends to a deterministic choice model, as described below). Third, the algorithm identifies the turn of the γ-fixed point and call the Binding procedure to select the most violated constraint at each zone.

A weakness of this algorithm is that it is constrained to zoning regulations restricted to the linear form. To relax this limitation the appropriate analysis is required to assure the convergence of the specific γ–fixed point; or a more general optimization methods should be used, but then the convergence properties depend on the specific problem and the method used.

3.3. Specification of the simulation test

The following conditions have been applied in the simulation. Bids of consumer agents and suppliers' profit functions are IID Gumbel distributed, with scale factors μ y λ respectively, such that the probabilities of location ($P_{h/vi}$, equation 3) and supply (S_{vi}, equation 8) are multinomial logit functions. Real estate profits are homogeneous in the industry, i.e. $\pi_j = \pi \ \forall j$.

Additionally, consumer bids are additive functions, like in (1). The fact that location probabilities are interdependent, is due to attributes that describe location externalities, which is specified in term b^2. In this paper, the following linear form is used:

$$b^2_{hvi} = \alpha_h \sum_{h'v'} Z_{h'} P_{h'/vi} S_{vi} + \beta_h \sum_{v'} Y_{v'i} S_{v'i} \tag{21}$$

The first term with $P_{h/vi}$ describes generic location attributes associated to the distribution of agents: the neighborhood quality related with the socioeconomic characteristics of other households in the zone and the value of agglomeration externalities given by presence of economic activities. Here Z describes agents characteristics such as average income of households, number of commercial businesses in the area, etc. The second term with $S_{v'i}$, describes the externalities associated to the built environment. In this case Y describes residential density, average building height, etc. With these two types of terms any set of (linearly defined) zone attributes can be specified with the model variables. The sets of parameters α and β represent values that the consumer assigns to each attribute of the zone, which is called the "hedonic price" and is calibrated from observations of locations and rents.

3.4. Sensitivity Analysis

For each fixed point, and also for the complete equilibrium set of equations (19), the functional form and convergence is studied through simulation and the sensitivity to their most relevant parameters. The dimensions used in the simulation are: 4 agents clusters (*h)*; 5 zones (*i*) and 2 dwelling types (*v*). A population of 100 agents is considered distributed in the following way:

Cluster	N° agents	Average Income (Z_h)
1	10	4 units.
2	15	3 units.
3	25	2 units.
4	50	1 units.

The information and parameters used in the simulation exercise are fictitious, though similar to those of the city of Santiago, Chile.

The main results obtained are:

- Dependence on the starting point. The solution is independent on the starting point, it only becomes dependent once the multinomial scale parameters μ and λ are large, which reflects a deterministic behavior of agents choice process; deterministic bids and profits respectively.
- Convergence: Individually, each function converges very fast, between two and six iterations, which indicates that the multinomial and logsum functions are contracting. The

system of fixed points has the same property. This result is highly relevant because allows to study equilibrium in large urban systems at a very low computational cost, despite the high complexity of the model.

- Sensitivity analysis. The solution is highly robust to marginal changes in the model parameters. However, unstable solutions appear for large scale parameters μ and λ, associated to deterministic behavior.
- Important parameter. The most important in the equilibrium solution are the scale parameters μ and λ. Residential locations externalities, represented by the average zone income have the strongest power to shift the location probability curves, thus affecting the solution, but it does not affect the curve form nor the convergence property.

The deterministic –all or nothing- behavior represented by large scale parameters, produces an unstability that is justified by the theory, since it means that the best bidder in the auction changes drastically for small changes in the supply attributes. This implies that the land use pattern is unstable and so are the attributes reflecting location externalities. In fact the deterministic the equilibria space contains multiple points.

Note that the solution does not give an absolute value for b^1, but relative ones, which defines the indeterminacy of the absolute values of bids and the existence of multiple solutions. Nevertheless, the solution space is unique and solutions differ only by b^3. This is precisely what is determined for uniqueness upon determining b^3 from the algorithm outputs.

4. Conclusions

The equilibrium model of the land market presented in this paper incorporates the idiosyncratic nature of supplier' behavior that, added to the random bidding model of consumer behavior previously developed, generates the new Random Bidding and Supply Model, RB&SM. This model is totally defined within the platform provided by the Gumbel distribution, so that the set of equations that define the equilibrium problem have a unique structure and there is statistical consistency among all the variables and equations of the model.

The logit supply model proposed has several advantages. It generates highly efficient fixed-point algorithms despite the complexity introduced. It is able to describe economies of scale and scope, usually syndicated as responsible of density tendencies in the real estate market. It also models the large number of zone regulations in the urban market, producing an output with an index of the economic impact of each regulation and their price effect in land values.

By means of simulations we have concluded that the RB&SM model finds unique solutions for a wide range of parameters including starting points and scale parameters, except when behavior becomes deterministic, i.e. when the variance of the random distributions decrease (scale factors increase). This is an expected result in non-linear models of discrete choice because the equivalent deterministic model is clearly path dependent: small changes in the choice of agents by the auction process generate further changes in the externalities that affect other auction results; Thus, the deterministic process is discontinuous and higher choice variance introduce the continuity in the equilibrium that generates stable and unique solutions.

The model can be extended in various aspects to relax the assumptions of the simulations studied here. One consists of introducing a process to generate the bidders choice set for the auctioneer, to change the assumption that all agents are potential bidders. A second aspect consists of introducing non-linear restrictions to equilibrium, for example to represent more complex urban regulations; this requires the use of other algorithms whose solutions are in general less stable.

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