3rd International Seminar on
Railway Operations Modelling and Analysis

RailZurich 2009

10-13 February 2009, Zurich

Railway Timetable Stability Analysis Using Stochastic Max-Plus Linear Systems

Rob M.P. Goverde, Bernd Heidergott, Glenn Merlet

r.m.p.goverde@tudelft.nl

Department Transport and Planning
Faculty of Civil Engineering and Geosciences

Delft University of Technology

Outline

Railway Timetable Stability Analysis Using Stochastic Max-Plus Linear Systems

- Introduction
- Stochastic max-plus linear systems
- Max-plus ergodic theory
- Stochastic stability analysis
- Example
- Conclusions

Introduction

Railway timetable stability

- The property that a timetable is able to recover from initial delays and primary delays (due to process time variations) without rescheduling
- How can stability performance be evaluated?

Issues

- Primary delays are unavoidable
- Secondary delays depend on primary delays and timetable
- Delay propagation of initial, primary, and secondary delays must be kept within bounds
- Complex problem depending on timetable constraints (regular intervals, synchronization, no early departures), interconnection structure, infrastructure constraints, rolling stock circulations
- Delay recovery by effective time supplements and buffer times

Stochastic max-plus linear systems

- Event time of the k-th occurrence of event i : $x_i(k)$
  - ❑ E.g. arrival time, departure time, passage time at any 'timetable point'
- Process time from event j to k-th occurrence of event i : $a_{ij}(k)$
  - ❑ E.g. running time, dwell time, transfer time, minimum headway time, turn-around time, ...
- An event occurs only if each preceding process from a predecessor event j has finished:

$$x_i(k) = \max_j (a_{ij}(k) + x_j(k-1)), \quad k \geq 1$$

- Let $a_{ij}(k) = -\infty$ if j is not a predecessor of i, then

$$x_i(k) = \max_{j=1,\ldots,n} (a_{ij}(k) + x_j(k-1)), \quad k \geq 1$$

Intermezzo

Max-plus algebra — Conventional algebra

- Define for real numbers and $-\infty$

$$a \oplus b = \max(a,b) \qquad a+b$$

$$a \otimes b = a+b \qquad a \times b$$

- Define for matrices

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij}) \qquad (A+B)_{ij} = a_{ij} + b_{ij}$$

$$(A \otimes B)_{ij} = \bigoplus_{l=1}^{n} (a_{il} \otimes b_{lj}) = \max_{l=1,\ldots,n} (a_{il} + b_{lj}) \qquad (AB)_{ij} = \sum_{l=1}^{n} (a_{il} \times b_{lj})$$

$$(c \otimes A)_{ij} = c \otimes a_{ij} = c + a_{ij} \qquad (cA)_{ij} = c \times a_{ij}$$

Stochastic max-plus linear systems

- Vector of k-th event times $x(k)$
- Matrix of process times $A(k) = (a_{ij}(k))$
- Then event times satisfy linear system equations in max-plus algebra:

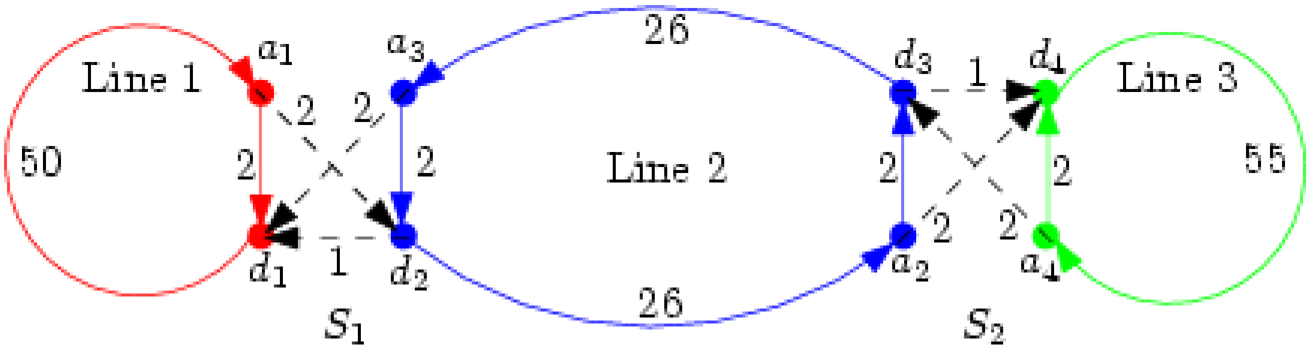
$$x(k) = A(k) \otimes x(k-1), \quad k \geq 1$$

- Note:

$$\begin{aligned} x_i(k) &= (A(k) \otimes x(k-1))_i \\ &= \bigoplus_{j=1}^{n} (a_{ij}(k) \otimes x_j(k-1)) \\ &= \max_{j=1,\ldots,n} (a_{ij}(k) + x_j(k-1)) \end{aligned}$$

- A random matrix A corresponds to a directed graph $G(A) = (V,E)$, with $V = \{1,\ldots,n\}$ and $E = \{(j,i) \mid a_{ij} \neq -\infty\}$, with random arc weights

Stochastic max-plus linear systems



$$
\begin{bmatrix} d_1(k) \\ d_2(k) \\ d_3(k) \\ d_4(k) \\ a_1(k) \\ a_2(k) \\ a_3(k) \\ a_4(k) \end{bmatrix}
=
\left[\begin{array}{cccc|cccc}
\cdot & 1 & \cdot & \cdot & 2 & \cdot & 2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 2 & \cdot & 2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & 2 \\
\cdot & \cdot & 1 & \cdot & \cdot & 2 & \cdot & 2 \\
\hline
50 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 26 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 26 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 55 & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
\otimes
\begin{bmatrix} d_1(k-1) \\ d_2(k-1) \\ d_3(k-1) \\ d_4(k-1) \\ a_1(k-1) \\ a_2(k-1) \\ a_3(k-1) \\ a_4(k-1) \end{bmatrix}
$$

Stochastic max-plus linear system

- Periodic timetable: vector of k-th scheduled event times $d(k)$

$$d(k) = d_0 \otimes T^{\otimes k} = (d_i(0) + k \cdot T)$$

with cycle time T and basic scheduled event times $d_i(0) \in [0, T)$

- The scheduled railway system satisfies

$$x(k) = A(k) \otimes x(k-1) \oplus d(k), \quad x(0) = x_0$$

with initial condition x_0: the initial event times at the start of the day

- The matrices $A(k)$ represent the primary process times which may generate primary delays when exceeding the scheduled process times
- The secondary delays are computed from the system equations when events have to wait for delayed preceding processes

Stochastic max-plus linear systems

Assumptions and properties

- An entry $a_{ij}(k)$ is either nonnegative or $-\infty$ for all k (fixed support)
- The finite entries $a_{ij}(k)$ are integrable nonnegative random variables (possibly dependent within the same period k)
- $\{A(k) \mid k \geq 1\}$ is a stationary or i.i.d. sequence of random matrices

- For simplicity: $A(k)$ is irreducible, i.e., $G(A(k))$ is strongly connected
- For simplicity: $A(k)$ has cyclicity 1

- A (scheduled) stochastic max-plus linear system

$$x(k) = A(k) \otimes x(k-1) \oplus d(k), \quad x(0) = x_0$$

is a stochastic event graph (stochastic decision-free Petri net)

Max-plus ergodic theory

- What is the behaviour of the event time sequence $\{x(k)\}_{k\geq 0}$, defined by the autonomous system (trains do not wait on timetable)

$$x(k) = A(k) \otimes x(k-1), \quad x(0) = x_0$$

- There exists a fixed cycle time λ, such that for each i and any $x_0 \geq 0$,

$$\lim_{k\to\infty} \frac{x_i(k)}{k} = \lim_{k\to\infty} \frac{\mathrm{E}[x_i(k)]}{k} = \lim_{k\to\infty} \mathrm{E}[x_i(k) - x_i(k-1)] = \lambda$$

- So the asymptotic behaviour is independent from the initial condition
- The value λ depends only on the structure and probability distribution of the random matrices $A(k)$ and is called its Lyapunov exponent

Stochastic stability analysis

- What is the behaviour of the event time sequence $\{x(k;T)\}_{k\geq 0}$, defined by the scheduled system

$$x(k) = A(k) \otimes x(k-1) \oplus d(k), \quad x(0) = x_0$$
$$d(k) = T \otimes d(k-1), \quad d(0) = d_0$$

- **Proposal**: A scheduled system is stable if for the primary process time distributions and any initial condition the cycle time equals T,

$$\lim_{k\to\infty} \frac{x_i(k;T)}{k} = T$$

- For each i and any $x_0 \geq 0$, the cycle time for the scheduled system is

$$\lim_{k\to\infty} \frac{x_i(k;T)}{k} = \lim_{k\to\infty} \frac{\mathrm{E}[x_i(k;T)]}{k} = \lambda \oplus T$$

- A scheduled railway system is stable iff $\lambda < T$

Stochastic stability analysis

- Delay sequence $\{z(k)\}$ is defined by

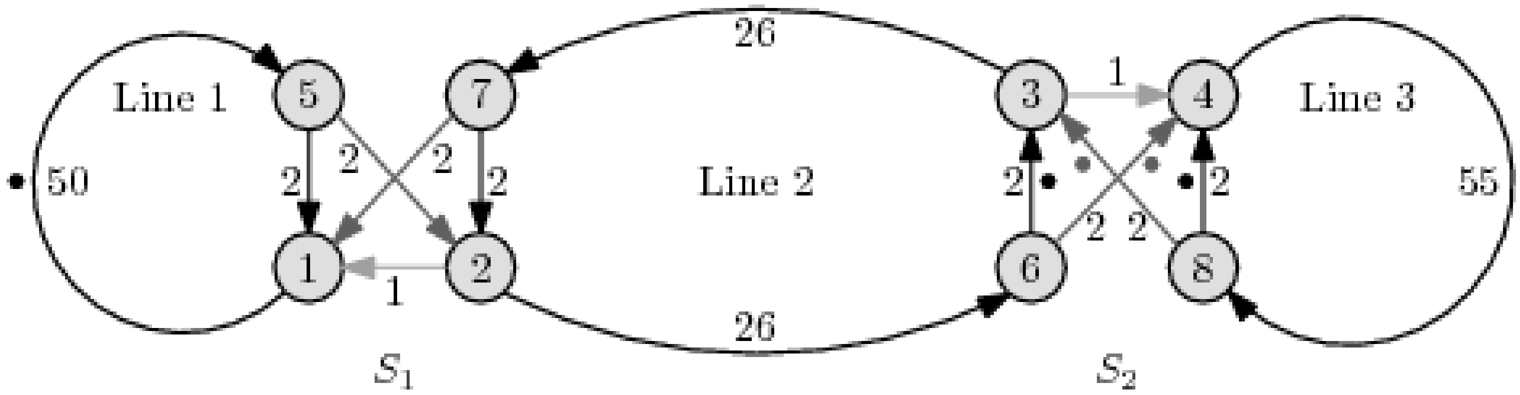
$$x(k) = A(k) \otimes x(k-1) \oplus d(k), \quad x(0) = x_0$$
$$d(k) = T \otimes d(k-1), \quad d(0) = d_0$$
$$z(k) = x(k) - d(k)$$

- **Proposal**: A timetable is realizable if for zero initial delays, $x_0 = d_0$, any delays generated by the primary process time distributions can settle,

$$\liminf_{k\to\infty} z(k; d_0) = 0$$

- Note: the delay sequence $z(k)$ will generally not converge to zero, since there will always be primary delays generating a new sequence of secondary delays
- Liminf implies that delays always settle, although new delays can occur

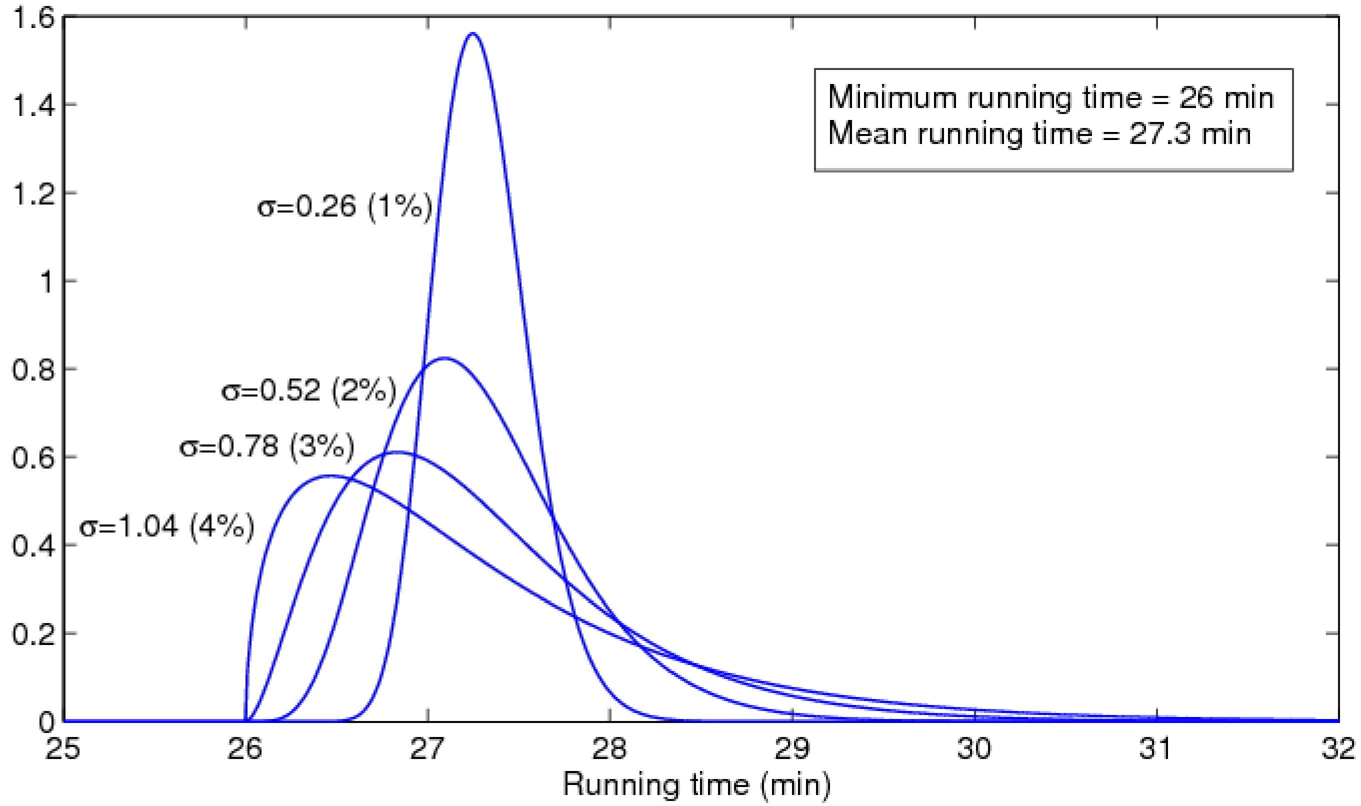
Example



- Periodic timetable: $d_0 = (31, 30, 0, 1, 21, 56, 26, 56)', T = 60$
- Primary process times are shifted Gamma distributed, where the shift is the minimum process time indicated in the figure
- The mean and standard deviation are given in percentage of the minimum process times, the Gamma parameters are estimated by matching moments

Èxample

- PDF of running time from station 1 to station 2



Example

- Cycle time as a function of mean and standard deviation as percentage of the minimum process times

	Standard deviation σ					
Mean μ	0%	1%	2%	3%	4%	5%
0%	58.0	-	-	-	-	-
1%	58.6	58.6	58.6	58.7	58.9	59.0
2%	59.2	59.2	59.2	59.3	59.5	59.6
3%	59.7	59.7	59.8	59.9	60.0	60.2
4%	60.3	60.3	60.3	60.4	60.6	60.8
5%	60.9	60.9	61.0	61.0	61.2	61.4

- The deterministic system becomes critical when process times are increased by 3.45%, random systems with this mean are unstable

Conclusions

- Timetable stability of large scale networks can be tested for arbitrarily distributed process times using stochastic max-plus stability analysis

- A fast algorithm based on perfect simulation has been developed for estimating the Lyapunov exponent of a given stochastic system
  - Primary process times can have arbitrary distributions (with finite mean)
  - Dependencies through cycles in the network are no problem

- Sensitivity analysis of distribution parameters gives insight in stability robustness