# Exact and Approximate Shortest-Path Queries for Planar Graphs

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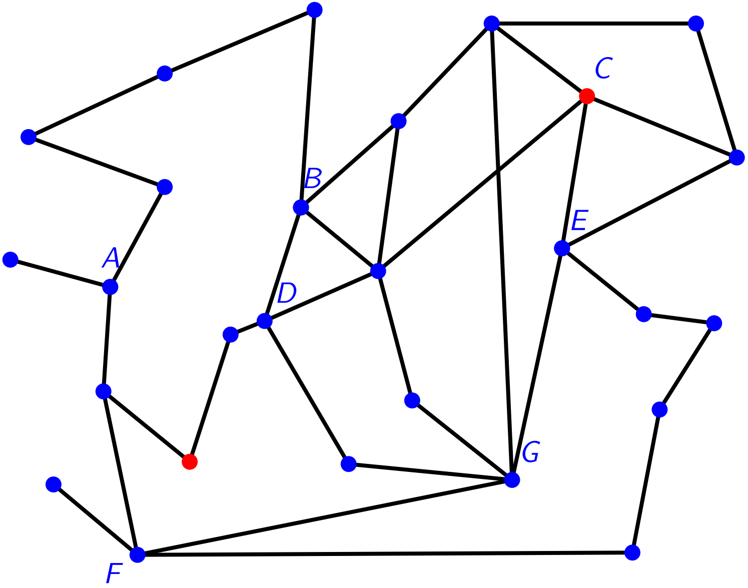
Joint work with Shay Mozes (exact q.) and
Philip N. Klein and Ken-ichi Kawarabayashi (approximate)

15 February 2011, NII Shonan Workshop

# Motivation



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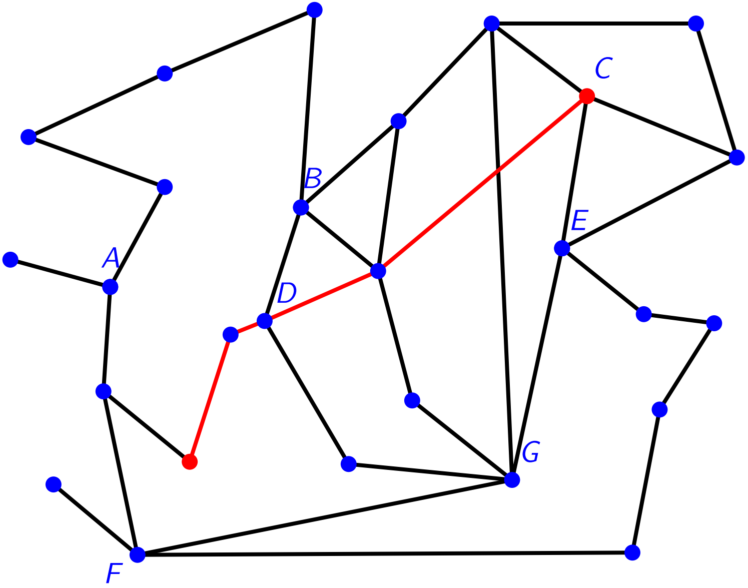

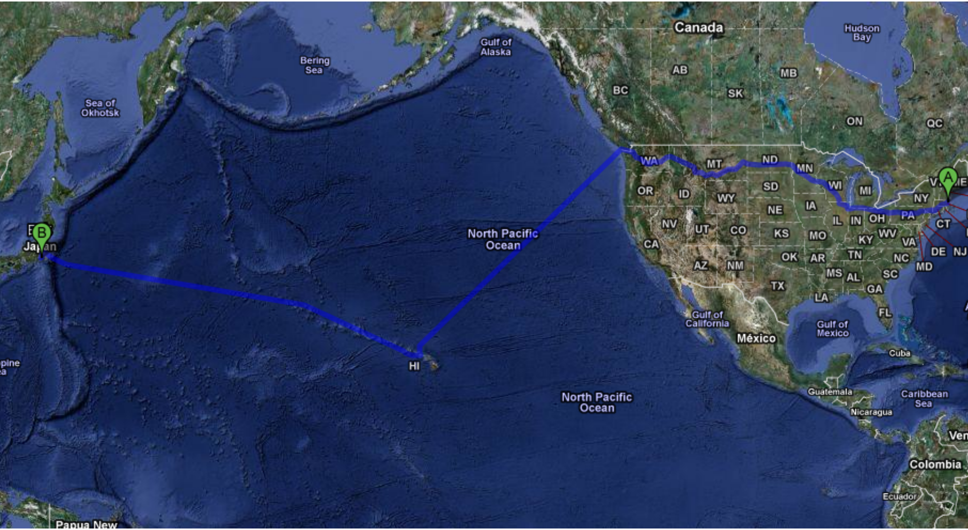

Canada
Hudson Bay
Gulf of Alaska
Bering Sea
Sea of Okhotsk
AB
MB
BC
SK
ON
QC
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OR
ID
WY
SD
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MS
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SC
MD
TX
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FL
A
B
North Pacific Ocean
HI
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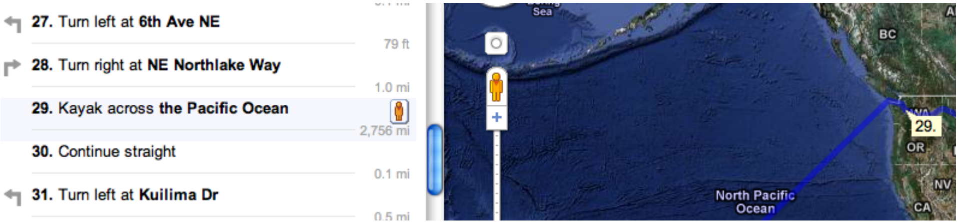
México
Gulf of Mexico
Cuba
North Pacific Ocean
Guatemala
Caribbean Sea
Nicaragua
Colombia
Ecuador
Papua New


27. Turn left at 6th Ave NE
79 ft
28. Turn right at NE Northlake Way
1.0 mi
29. Kayak across the Pacific Ocean
2,756 mi
30. Continue straight
0.1 mi
31. Turn left at Kuilima Dr
0.5 mi
Sea
BC
29.
OR
NV
CA
North Pacific
Ocean

# Shortest-Path Queries / Distance Oracles

- Preprocess a graph $G$ with $n$ nodes and $m$ edges ...
- ... to create a Data Structure, using which ...
- ... we can efficiently answer Distance Queries.
  - $d(u, v)$

# Shortest-Path Queries / Distance Oracles

- Preprocess a graph $G$ with $n$ nodes and $m$ edges ...
- ... to create a Data Structure, using which ...
- ... we can efficiently answer Approximate Distance Queries.
    - $\widetilde{d}(u, v)$

# Approximate Distance Oracles — Stretch

- Distance between $u$ and $v$ in graph $G$: $d_G(u, v)$
- Oracle Result $\widetilde{d}(u, v)$ satisfies

$$d_G(u, v) \leqslant \widetilde{d}(u, v) \leqslant \alpha \cdot d_G(u, v).$$

- Multiplicative Stretch $\alpha$

# Related Work

Practical

- Focus on Transportation Networks

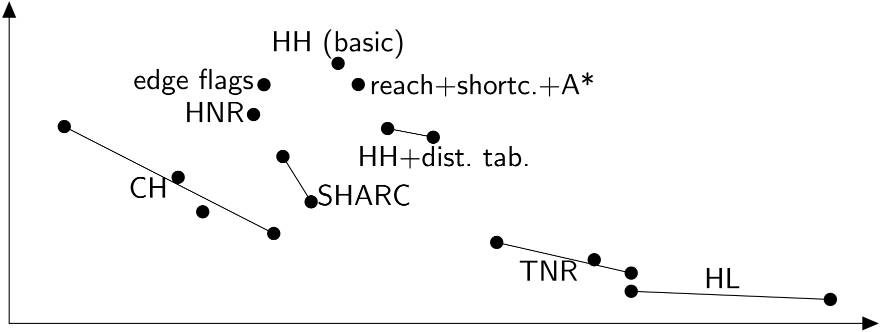
Theoretical

- General, undirected graphs
- Restricted classes (planar, bounded tree-width, minor-closed,...)

# Shortest-Path Queries in Transportation Networks

- Main focus, large body of research since 60's/70's
- Big progress around 2006 (DIMACS Implementation Challenge)
  - Preprocessing: tens of minutes for road map of the US/EU
  - Query time: $\approx 10^6$ times faster than Dijkstra's algorithm
- Ideas
  - Geometry, coordinates, A* search [SV86]
  - Goal-directed search (A* for graphs) [GH05]
  - Hierarchical structures [SS05, BFSS07, BD08, BDS$^+$08]
- Heuristics that work very well for road networks (separators)
- Visit only edges of shortest path

# Space vs. Query Time Tradeoffs: Experimental Results



# Related Work

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- General, undirected graphs
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# Outline

- Introduction
- Planar Separators
- Approximate Distance Oracles
- Exact Distance Oracles
- Conclusion

# Planar Separators, Graph $G = (V, E)$

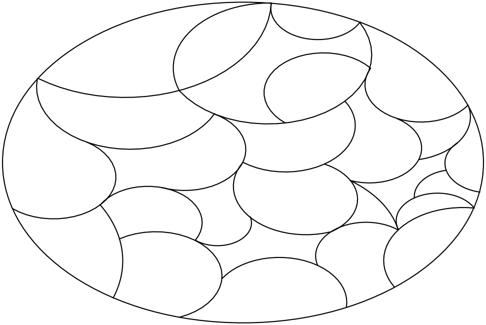
Partition $V$ into $V_1, V_2, S$
such that $|V_1|, |V_2| \leqslant \frac{n}{2}$, no edge between $V_1$, $V_2$, and

- Lipton and Tarjan [LT80], Miller [Mil86]
  - s.t. $|S| = \mathcal{O}(\sqrt{n})$
  - Shortest paths may cross $S$ up to $\mathcal{O}(\sqrt{n})$ times
- Thorup [Tho04]
  - s.t. $S$ consists of 3 shortest paths
  - can be extended to minor-closed families [AG06]
- Dieng and Gavoille [DG09]
  - s.t. $S$ consists of $\mathcal{O}(1)$ shortest paths of length "tree-length"

## $r$–divisions [Fre87]

separate recursively (e.g. using [Mil86]) into

- $O(n/r)$ regions
- region size $O(r)$
- region boundary $O(\sqrt{r}) \rightsquigarrow$ total boundary $O(n/\sqrt{r})$



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# $(1+\epsilon)$–Approximate Shortest-Path Queries; Planar $G$

| Preprocessing | Space | Query | Reference |
|---|---|---|---|
| $O(n\epsilon^{-2}\lg^4 n)$ | $O(n\cdot\epsilon^{-1}\lg^2 n)$ | $O(\epsilon^{-1}+\lg\lg n)$ | [Tho04, Thm.3.16] |
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| $O(n\epsilon^{-2}\lg^4 n)$ | $O(n)$ | $O(\epsilon^{-2}\lg^3 n)$ | NEW |
| $O(n\epsilon^{-2}\lg^3 n)$ | $O(n\cdot\epsilon^{-1}\lg n)$ | $O(\epsilon^{-1})$ | [Tho04, Thm.3.19] |
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Assumption for this table: largest integer weight $N = O(poly(n))$
(complexity of oracles for planar digraphs depends on $N$)

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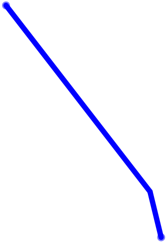
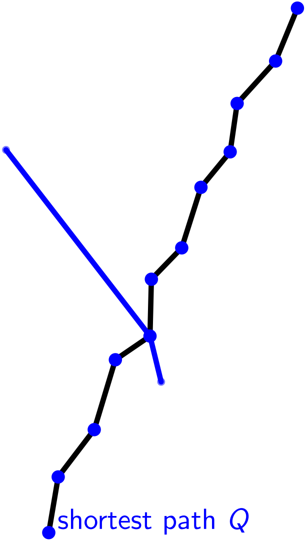
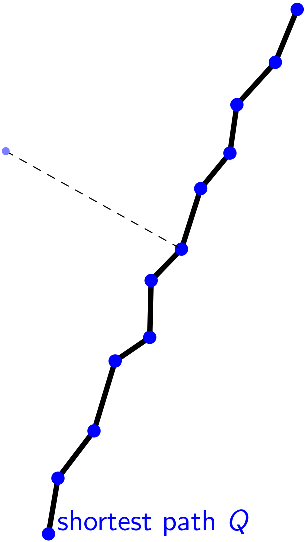
# Main Techniques of Thorup's Distance Oracle

- Partition $V$ into $V_1, V_2, S$ such that $|V_1|, |V_2| \leqslant \frac{n}{2}$ and
  - s.t. $S$ consists of 3 shortest paths $Q$
  - $\rightsquigarrow$ shortest paths cannot cross many times
- Representation of paths that intersect $Q$




shortest path $Q$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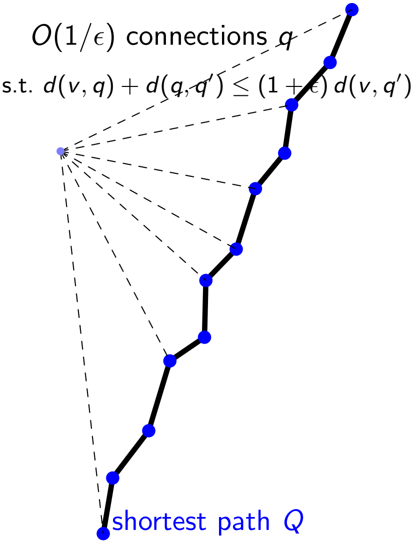
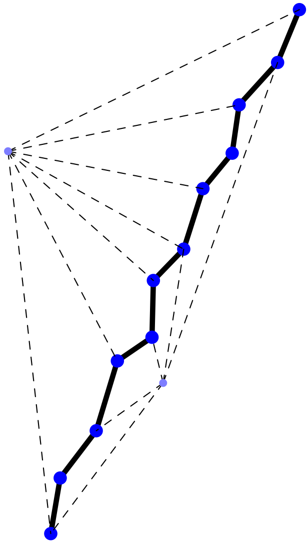

$O(1/\epsilon)$ connections $q$
s.t. $d(v, q) + d(q, q') \leq (1 + \epsilon)\, d(v, q')$
shortest path $Q$



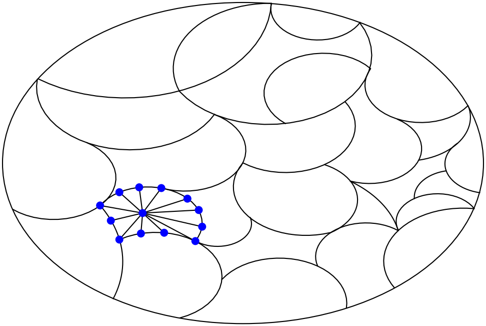
# Space Consumption of Thorup's Distance Oracle

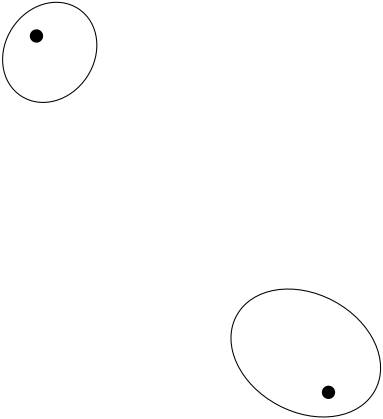
- Recursive partition using 3 shortest paths $Q$ per level
  - $O(\log n)$ shortest-path separators per node
- Representation of paths that intersect $Q$
  - store $O(1/\epsilon)$ connections
- Total storage: $O(\epsilon^{-1} \log n)$ connections per node

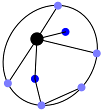
# Linear-Space Distance Oracle: Main Idea

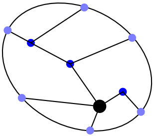
- store connections for few nodes (landmarks)
  $\rightsquigarrow$ boundary of $r$–division!

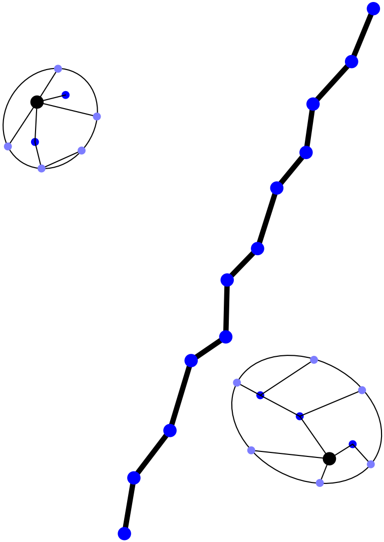
- at query time, search landmark then use [Tho04]

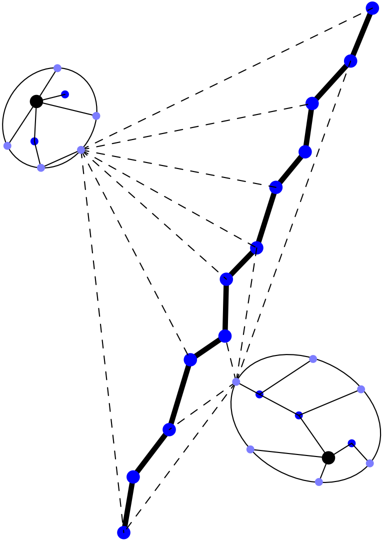


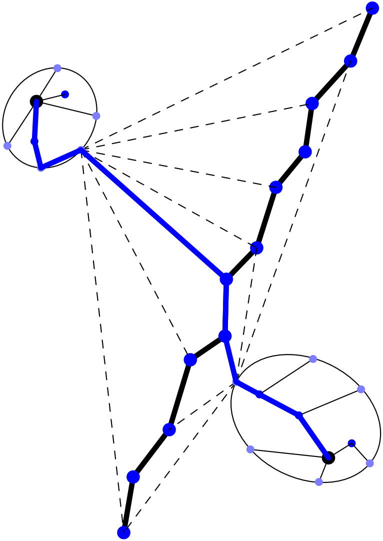


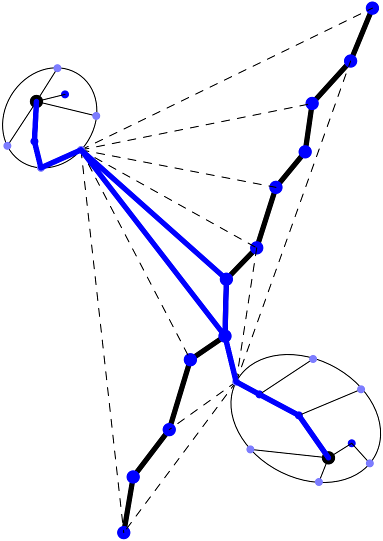


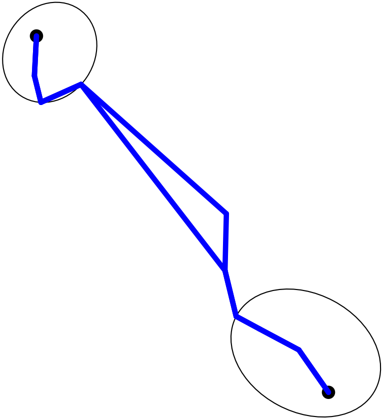












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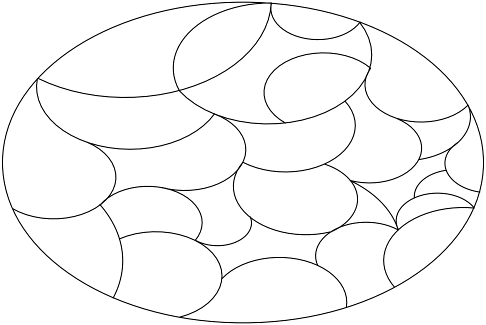
# Outline

## $r$–divisions [Fre87]

separate recursively (e.g. using [Mil86]) into

- $O(n/r)$ regions
- region size $O(r)$
- region boundary $O(\sqrt{r}) \rightsquigarrow$ total boundary $O(n/\sqrt{r})$



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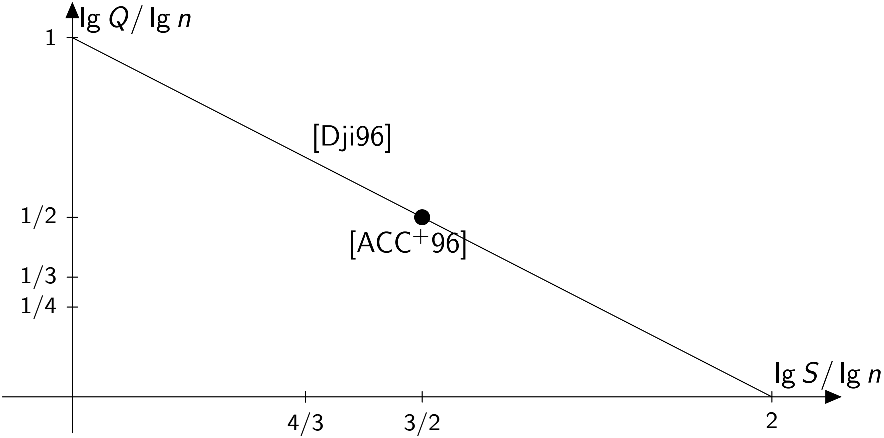
Idea: store connections among boundary nodes

- space $O(n^2/r)$
- query time $O(r)$ (explore region, compute min boundary pair)

# Space vs. Query Time for Exact Shortest Paths



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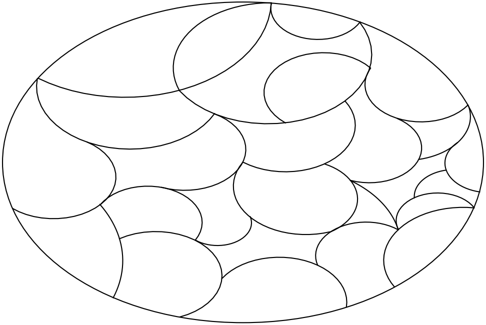
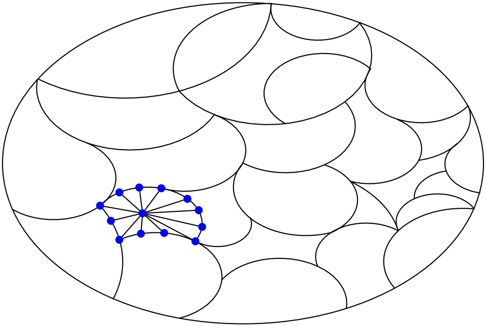


# How To Use $r$–divisions [Dji96]



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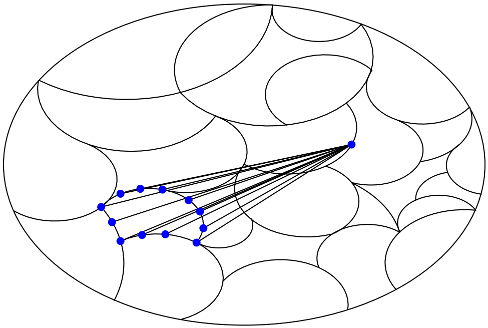

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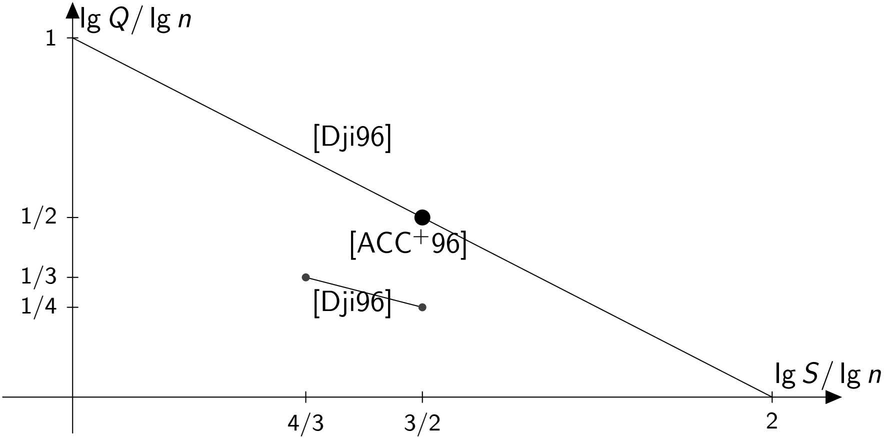
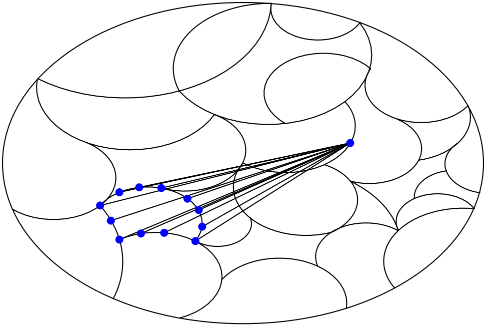


# Planarity-Exploiting Search at Query Time [Dji96]



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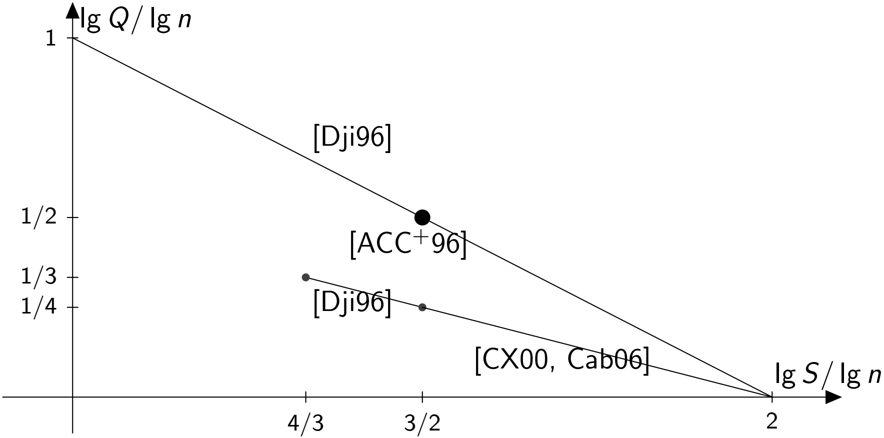

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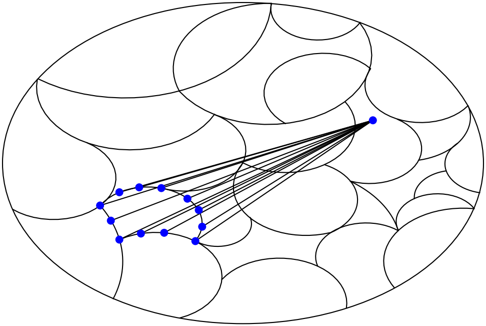

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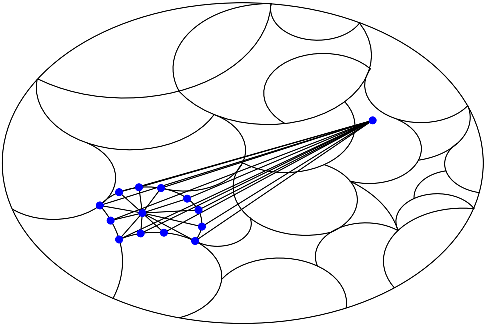
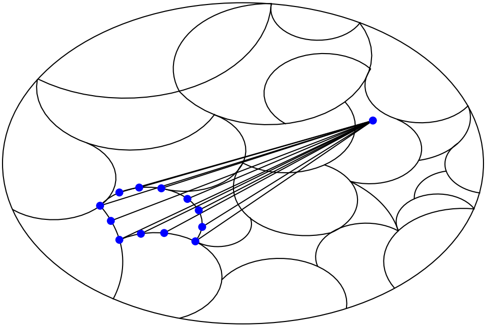


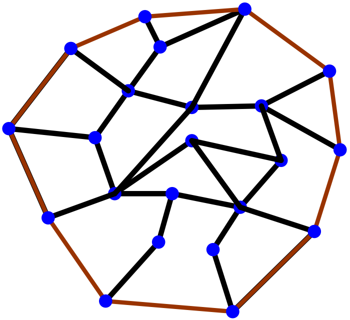
# Efficient Representation of Connection to Boundary



# Klein's MSSP data structure [Kle05]

preprocess $G, f$

$O(n \lg n)$ time & space



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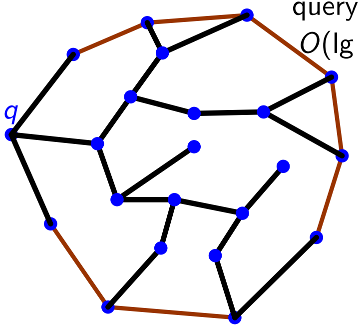


query $d(q, v)$
$O(\lg n)$ time

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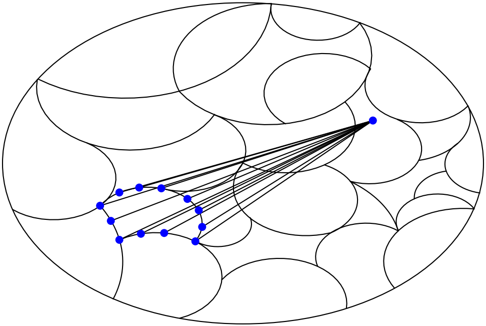

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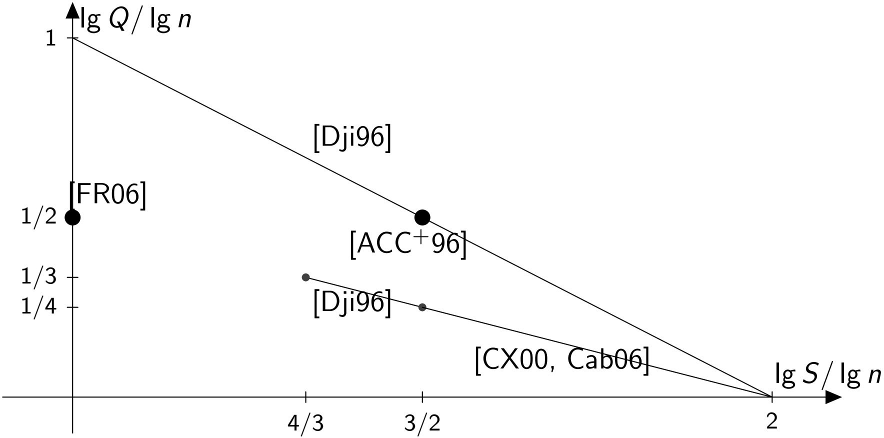

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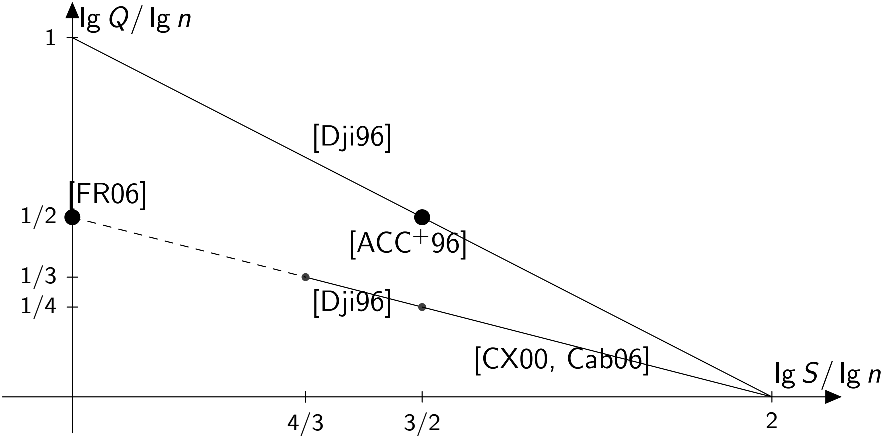

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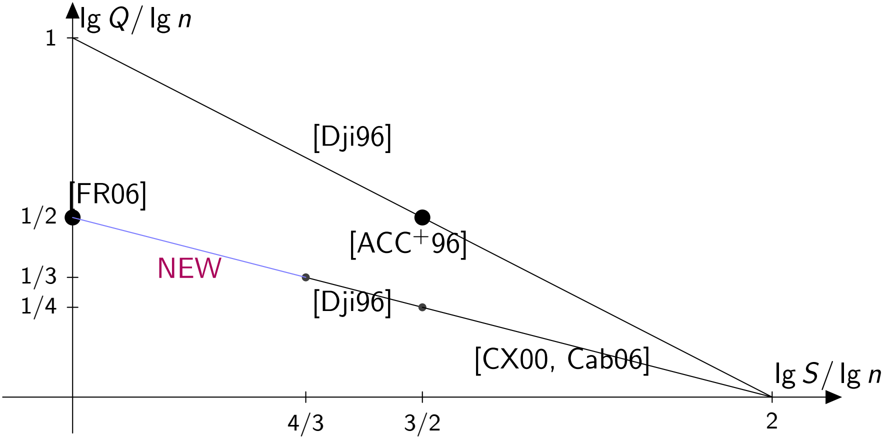
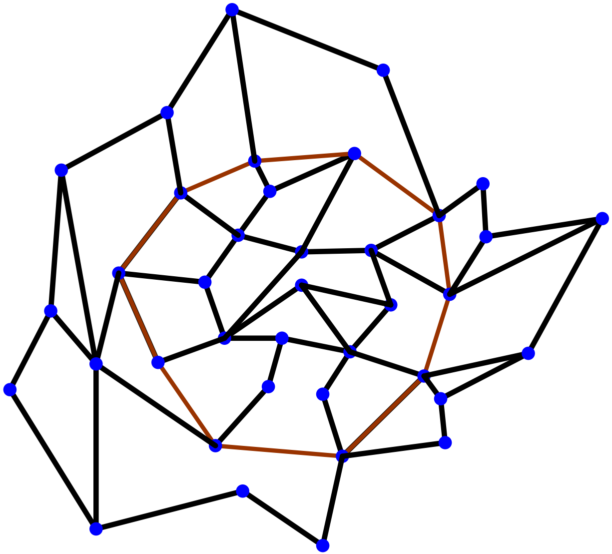


# New Cycle-MSSP data structure



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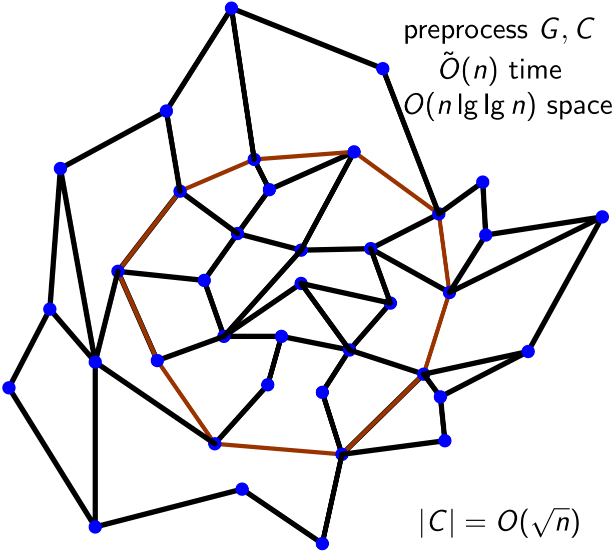
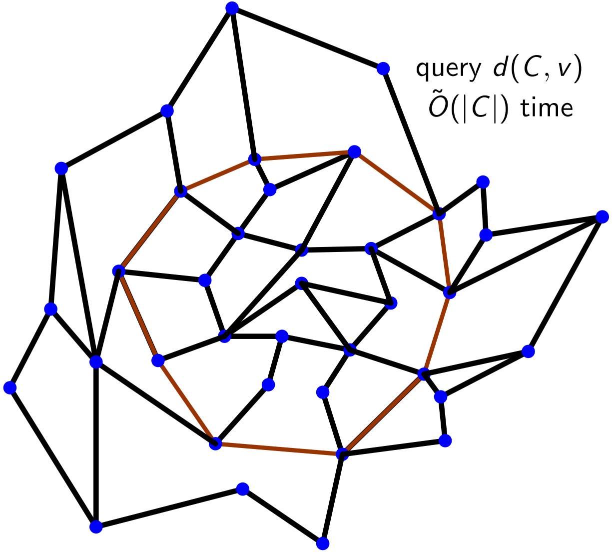


preprocess $G, C$
$\tilde{O}(n)$ time
$O(n \lg\lg n)$ space

$|C| = O(\sqrt{n})$

# New Batched Cycle-MSSP data structure



query $d(C, v)$

$\tilde{O}(|C|)$ time

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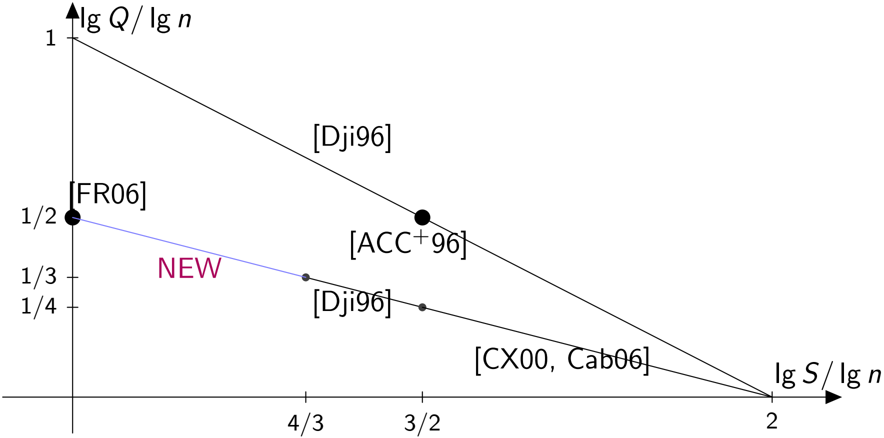
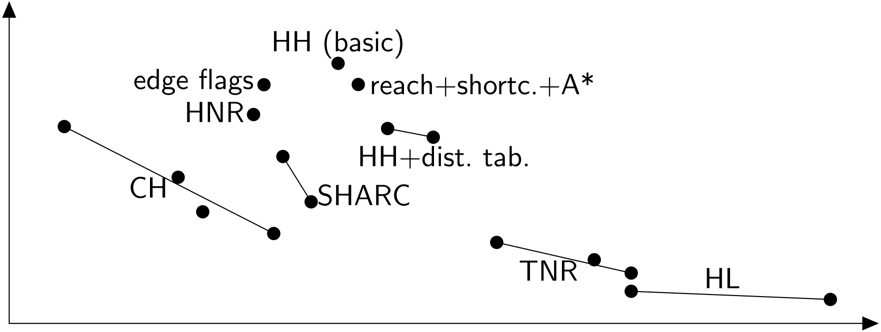


# Space vs. Query Time Tradeoffs: Experimental Results



# An Experimental Result...

Hilger et al. [HKMS09, Section 6]

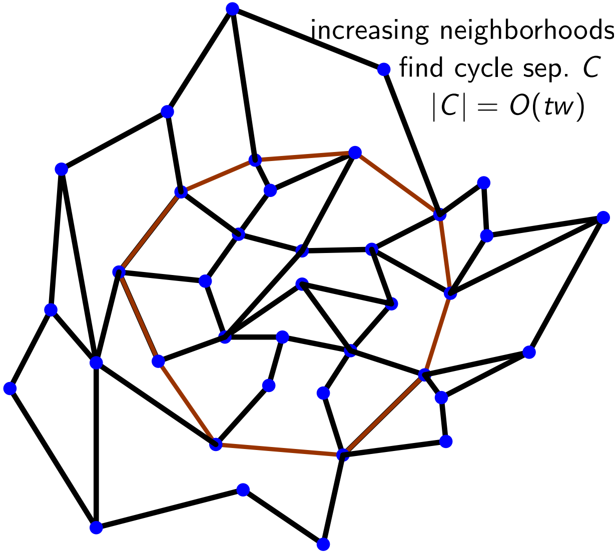
> *In all cases, the search space of our arc-flag method is never larger than ten times the actual number of nodes on the shortest paths.*

# ... and the Corresponding Theoretical Result

Can prove that

> *In* all *cases, the search space of our method is never larger than ~~ten~~ $O(\lg^2 \ell)$ times the ~~actual number of nodes on~~ length $\ell$ of the shortest paths.*

# Query Time Proportional to Path Length: Brief Sketch



# Outline

- Introduction
- Planar Separators
- Approximate Distance Oracles
- Exact Distance Oracles
- Conclusion

# Contributions and Outlook

- Query Time (quasi-) proportional to Shortest-Path Length (almost matches experimental results)
- Application determines how much space $S \geq m$, our tradeoffs tell how to use it for fast query time $Q$ both for approximate and exact
- Main open question: optimal use?

| | | |
|---|---|---|
| Exact | $S \cdot Q \geq n\sqrt{n}$ | ? |
| $(1+\epsilon)$–Approximate | $S \cdot Q \geq n \lg n$ | ? |
| $O(1)$–Approximate | $S \cdot Q \geq n \lg \lg n$ | ? |

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